Mathematical Models

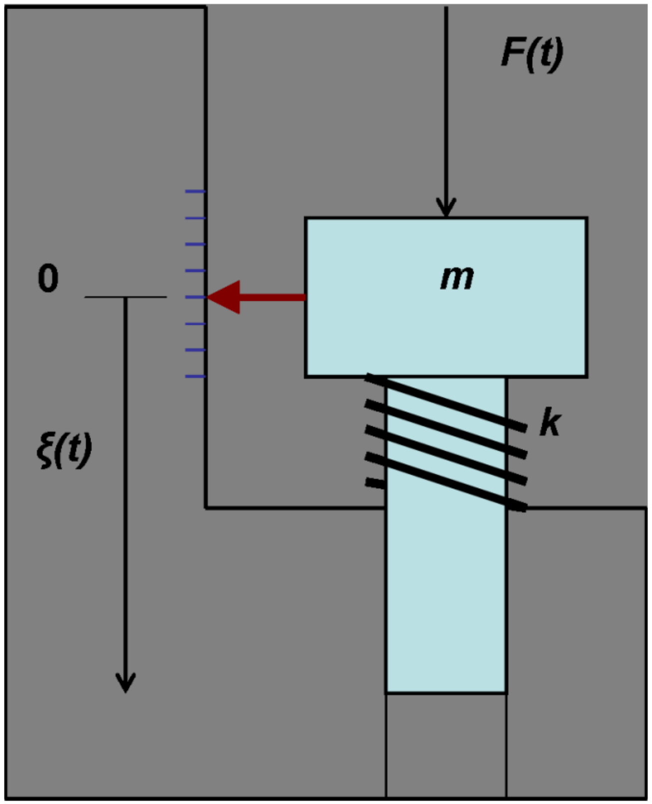
It is important to note that most real-world systems are too complex (nonlinear, stochastic, non-stationary, spatially distributed, etc.—recall the last lecture) to be modeled in every detail.

Mathematical models in general represent considerably simplified descriptions of the systems that they purport to model.

These simplifications are based on certain assumptions that need to be carefully inspected and if possible justified.

It is reasonable to start with a very simple model and add more complex details as necessary.

Example 1 Spring-mass system (scale)



system: scale (spring & mass)

input: forces $F(t)$ acting upon the mass

output: ξ

parameters: m [kg] and k [N/m]

states: (will see in a minute)

Let 0 be the position of the red hand, when $F(t) = 0$. Define ξ as a deviation from 0.

What happens if I step on the scale?

$$F(t) = \underbrace{M}_{\text{my mass}} g = \underbrace{F}_{\text{weight}}$$

Static equilibrium: $F_s = F$, where $F_s = k\xi$ (spring force—Hooke's Law).

Note: $g = 9.81$ [m/s^2] is the gravity acceleration.

Note: $k > 0$ [N/m] is the spring constant (stiffness).

Therefore $k\xi = F$, which implies that the mass will move from $\xi = 0$ to $\xi = F/k$.

Essentially, $\xi = F/k$ represents the mathematical model of the scale (albeit oversimplified):

$$\underbrace{y(t)}_{\text{output}} = \frac{1}{k}\underbrace{u(t)}_{\text{input}}$$

It belongs to the category of static models (see the last lecture—also try to identify the other properties).

But what if I am interested in how the mass gets from $\xi = 0$ to $\xi = F/k$? In other words, how do we describe ξ as a function of time?

Answering this questions (and similar ones) is the subject of <u>dynamic modeling</u>.

Dynamic balance of forces ("dynamic equilibrium"):

$$\underbrace{m\ddot{\xi}(t)}_{\text{inertia}} + \underbrace{k\xi(t)}_{\text{spring}} = \underbrace{F}_{\text{weight}}$$

Demo: `spring_mass_simulation.m`

Conclusion: this model does not correspond to the reality (something fundamental is missing).

Add friction forces (where $b > 0$ [kg/s] is the friction coefficient):

$$\underbrace{m\ddot{\xi}(t)}_{\text{inertia}} + \underbrace{b\dot{\xi}(t)}_{\text{friction}} + \underbrace{k\xi(t)}_{\text{spring}} = \underbrace{F}_{\text{weight}}$$

Demo: `spring_mass_simulation.m`

In reality, both the friction coefficient and spring constant are nonlinear, e.g.

$$\underbrace{m\ddot{\xi}(t)}_{\text{inertia}} + \underbrace{b(\dot{\xi})\dot{\xi}(t)}_{\text{friction}} + \underbrace{k(\xi)\xi(t)}_{\text{spring}} = \underbrace{F}_{\text{weight}}$$

but this would be overkill.

In general, a simpler model may be common to many seemingly different problems, whereas highly “tuned” models are only applicable to a single problem (overfitting).

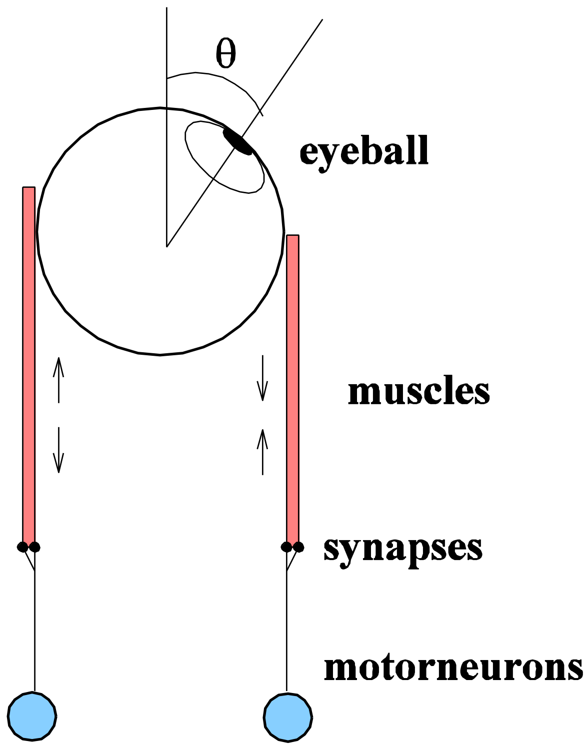
Simple models are especially important because they can often be treated using analytical tools, and could be valuable in the development of new theories.

Complex models typically give rise to simulations (computational science).

Balancing the complexity and usefulness is in itself a research problem.

Input-Output and State-Space Models

Example 2 Eye movement model (note the analogy to the spring-mass system)



$$J\ddot{\theta}(t) + B\dot{\theta}(t) + K\,\underbrace{\theta(t)}_{\text{output}} = \underbrace{\tau(t)}_{\text{input}}$$

$$J, B, K \; - \text{parameters}$$

Left hand side: output variable and its derivatives

Right hand side: input variable (and its derivatives)

This type of model is called the <u>input-output model</u> of a system.

In general, for linear time-invariant (LTI) systems, the input-output model takes the following form:

$$\boxed{\sum_{k=0}^{n} a_k y^{(k)}(t) = \sum_{k=0}^{p} b_k u^{(k)}(t)} \qquad (1)$$

where $n \geq p, a_n = 1$ and $z^{(k)}(t) := d^k z(t)/dt^k$.

The <u>order of the system</u>, n, is equal to the order of the highest derivative on the left hand side of (1).

In the spirit of systems theory, high-order ODEs are converted to a first-order ODE, i.e. the input-output model (1) can be written in the form:

$$\boxed{\begin{array}{ll} \dot{x}(t) = Ax(t) + Bu(t) & - \text{state equation} \\ y(t) = Cx(t) + Du(t) & - \text{output equation} \end{array}}$$

state-space model of an LTI system

The model above is called the <u>state-space model</u>.

This model is seemingly simpler than the one given by (1) (note first order ODE vs. n-th order ODE).

The trick is that x is now a vector in $\mathbb{R}^n$. In particular (if $p = 0$), x can be taken in a simple form:

$$\left.\begin{aligned} x_1(t) &:= y(t) \\ x_2(t) &:= \frac{dy(t)}{dt} \\ &\vdots \\ x_n(t) &:= \frac{d^{n-1}y(t)}{dt^{n-1}} \end{aligned}\right\} \text{phase variables}$$

The variables $x_1(t), \cdots, x_n(t)$ represent the state variables, and the vector $x \in \mathbb{R}^n$ defined as:

$$x(t) := \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

is called the state vector.

The choice of the state variables is not unique, and the scheme above represents one canonical choice. Going back to the eye movement system:

$$\underbrace{1}_{a_2}\underbrace{\ddot{\theta}(t)}_{\ddot{y}(t)} + \underbrace{\frac{B}{J}}_{a_1}\underbrace{\dot{\theta}(t)}_{\dot{y}(t)} + \underbrace{\frac{K}{J}}_{a_0}\underbrace{\theta(t)}_{y(t)} = \underbrace{\frac{1}{J}}_{b_0}\underbrace{\tau(t)}_{u(t)}$$

This is just the eye movement input-output model written in the form of (1). Apply the scheme for choosing the state variables (note $n = 2$):

$$x_1(t) \coloneqq y(t)$$
$$x_2(t) \coloneqq \frac{dy(t)}{dt}$$

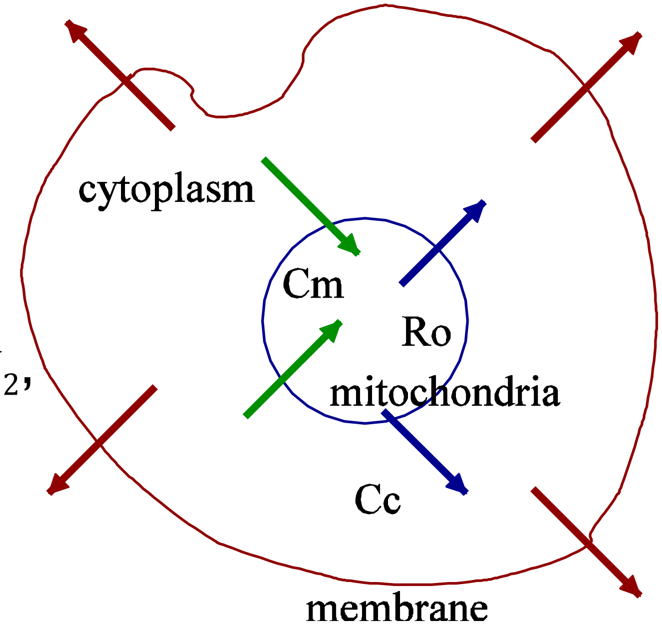
to obtain*:

$$\underbrace{\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix}}_{\dot{x}(t)} = \underbrace{\begin{bmatrix} 0 & 1 \\ -K/J & -B/J \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} 0 \\ 1/J \end{bmatrix}}_{B} u(t)$$

$$y(t) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{0}_{D} u(t)$$

Example 3 Cellular dynamics

system: cell
input: the rate of synthesis R_0
output: the concentration C_c
states: the concentrations C_m and C_c
parameters: the rate constants K_{12} and K_2, volumes V_m and V_c



$$V_m\dot{C}_m(t) = R_0(t) - K_{12}[C_m(t) - C_c(t)]$$
$$V_c\dot{C}_c(t) = K_{12}[C_m(t) - C_c(t)] - K_2 C_c(t)$$

The system is already in the state-space form:

$$\dot{C}_m(t) = -\frac{K_{12}}{V_m}\underbrace{C_m(t)}_{x_1(t)} + \frac{K_{12}}{V_m}\underbrace{C_c(t)}_{x_2(t)} + \frac{1}{V_m}\underbrace{R_0(t)}_{u(t)} \quad \text{– 1st state equation}$$
$$\dot{C}_c(t) = \frac{K_{12}}{V_c}C_m(t) - \left[\frac{K_{12}}{V_c} + \frac{K_2}{V_c}\right]C_c(t) \quad \text{– 2nd state equation}$$

$$\underbrace{\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix}}_{\dot{x}(t)} = \underbrace{\begin{bmatrix} -K_{12}/V_m & K_{12}/V_m \\ K_{12}/V_c & -(K_{12}+K_2)/V_c \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} 1/V_m \\ 0 \end{bmatrix}}_{B} u(t)$$

output equation $y(t) := C_c(t) = x_2(t)$ can be written in a matrix form:

$$y(t) = \underbrace{[0 \quad 1]}_{C} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{0}_{D} u(t)$$

Note: input-output and state-space models are not strictly tied to linear systems.

In particular, they can be defined for any type of system as long as the system is dynamic (static systems do not have states).

Example 4

Simplified model of neural dynamics
(H.R. Wilson. *J. Theoret. Biol.*, 1999)

$$C\frac{dV(t)}{dt} = -\underbrace{m_\infty(V)[V(t)-0.5]}_{\text{sodium current}} - \underbrace{26R(t)[V(t)+0.95]}_{\text{potassium current}} + \underbrace{I_{in}(t)}_{\text{inj. current}}$$

$$\frac{dR(t)}{dt} = \frac{1}{\tau_R}[-R(t) + R_\infty(V)]$$

where, V is the potential across the membrane, R is the conductance of the potassium channels, and

$$m_\infty(V) = 17.8 + 47.6V + 33.8V^2$$
$$R_\infty(V) = 1.24 + 3.7V + 3.2V^2$$

Define: $x_1 = V$, $x_2 = R$ (states); $u = I_{in}$ (input); $y = V$ (output)

$$\frac{dx_1(t)}{dt} = -\frac{1}{C}[17.8 + 47.6x_1(t) + 33.8x_1^2(t)][x_1(t) - 0.5] - \frac{26}{C}x_2(t)[x_1(t) + 0.95] + \frac{1}{C}u(t)$$

$$\frac{dx_2(t)}{dt} = \frac{1}{\tau_R}[1.24 + 3.7x_1(t) + 3.2x_1^2(t) - x_2(t)]$$

$$y(t) = \underbrace{[1 \quad 0]}_{C} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{0}_{D} u(t)$$

Note: the state space model of a nonlinear system *cannot* be written in the form

$dx(t)/dt = Ax(t) + Bu(t)$ and/or $y(t) = Cx(t) + Du(t)$

For nonlinear systems a more general notation is used:

$$\boxed{\begin{aligned} \dot{x}(t) &= f(x(t), u(t), t) && \text{-state equation} \\ y(t) &= g(x(t), u(t), t) && \text{-output equation} \end{aligned}}$$

state-space model of a nonlinear system

Linear Time-Invariant (LTI) Systems

Linearity

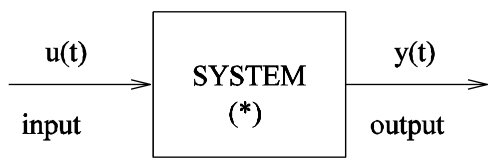
A function $f : \mathbb{R} \to \mathbb{R}$ is linear if

(1) $f(x+y) = f(x) + f(y) \quad \forall x, y \in \mathbb{R}$ (additivity)

(2) $f(ax) = af(x) \quad \forall a, x \in \mathbb{R}$ (homogeneity)

(1) and (2) can be combined into a single condition:

$f(ax+by) = af(x) + bf(y) \quad \forall a, b, x, y \in \mathbb{R}$ (superposition)



We saw last time that (*) can be anything. Using abstract notation (operator theory), we write

$$y(t) = \underbrace{L\left[t; t_0, x(t_0), u_{[t_0,t]}\right]}_{\text{function of a function}}$$

Think of L as a mapping $L : u \to y$, or $L : U \to Y$, where U and Y are sets of (admissible) inputs and outputs, respectively.

Example 5 Show that for the following system (assume $y(t_0)$ known):

$$\dot{y}(t) = u(t)$$

the operator L is given by*:

$$y(t) = L\left[t; t_0, x(t_0), u_{[t_0,t]}\right] = x(t_0) + \int_{t_0}^{t} u(\tau)d\tau$$

Hint: $\frac{\partial}{\partial t}\int_{u(t)}^{v(t)} f(t,\tau)d\tau = f\big(t, v(t)\big)\dot{v}(t) - f\big(t, u(t)\big)\dot{u}(t) + \int_{u(t)}^{v(t)} \frac{\partial}{\partial t} f(t,\tau)d\tau$

We will see later that L (for linear systems) can be expressed through a <u>convolution</u> operator.

Back to the definition of linearity.

Systems theory inherits the definition from mathematics:

Definition 1 *The system* $(*)$ *is linear if and only if the mapping* L *is linear with respect to the initial state* $x(t_0)$ *and the input* u.

1) Additivity

$$L\big[t; t_0, \bar{x}(t_0) + \bar{\bar{x}}(t_0), \bar{u}_{[t_0,t]} + \bar{\bar{u}}_{[t_0,t]}\big] = L\big[t; t_0, \bar{x}(t_0), \bar{u}_{[t_0,t]}\big] + L\big[t; t_0, \bar{\bar{x}}(t_0), \bar{\bar{u}}_{[t_0,t]}\big]$$
$$\forall \bar{u}, \bar{\bar{u}} \in \mathcal{U}, \qquad \forall \bar{x}(t_0), \bar{\bar{x}}(t_0) \in \mathbb{R}^n$$

2) Homogeneity:

$$L\big[t; t_0, \alpha x(t_0), \alpha u_{[t_0,t]}\big] = \alpha L\big[t; t_0, x(t_0), u_{[t_0,t]}\big]$$
$$\forall \alpha \in \mathbb{R}, \qquad \forall u \in \mathcal{U}$$

If L is the operator corresponding to the system (*) and if L satisfies (1) and (2), we say that (*) is a linear system!

(1) and (2) can be combined into a single condition—the superposition principle.

Definition 2 *A system L is linear if and only if it satisfies the superposition principle*.

Example 6 Show that the system: $\dot{y}(t) = u(t)$ is linear*

However, testing the linearity of a system by checking the additivity and homogeneity of its L operator is far from practical.

For us, the practical test of linearity will be the equation itself.

Example 7 Input-output model: $J\ddot{\theta}(t) + B\dot{\theta}(t) + K\theta(t) = \tau(t)$

Is this a linear ODE? Yes! Therefore, the system is linear. Alternatively look at the state-space model. Can you write the system in the form:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \quad -\text{state equation} \\ y(t) &= Cx(t) + Du(t) \quad -\text{output equation} \end{aligned}$$

so that the matrices A, B, C, and D do not depend on x and u ? Yes!

$$A = \begin{bmatrix} 0 & 1 \\ -K/J & -B/J \end{bmatrix}; B = \begin{bmatrix} 0 \\ 1/J \end{bmatrix}; C = \begin{bmatrix} 1 & 0 \end{bmatrix}; D = 0$$

Therefore, the system is linear!

Note: the system (*) need not be described by a (system of) ODE(s).

Example 8 The diffusion of oxygen in a living tissue

$$\frac{\partial y(\xi,\eta,\zeta,\mathrm{t})}{\partial t} = D\left(\frac{\partial^2 y(\xi,\eta,\zeta,t)}{\partial \xi^2} + \frac{\partial^2 y(\xi,\eta,\zeta,t)}{\partial \eta^2} + \frac{\partial^2 y(\xi,\eta,\zeta,t)}{\partial \zeta^2}\right) - ky(\xi,\eta,\zeta,t)$$

Is this a linear PDE? Yes! Therefore, the system is linear.

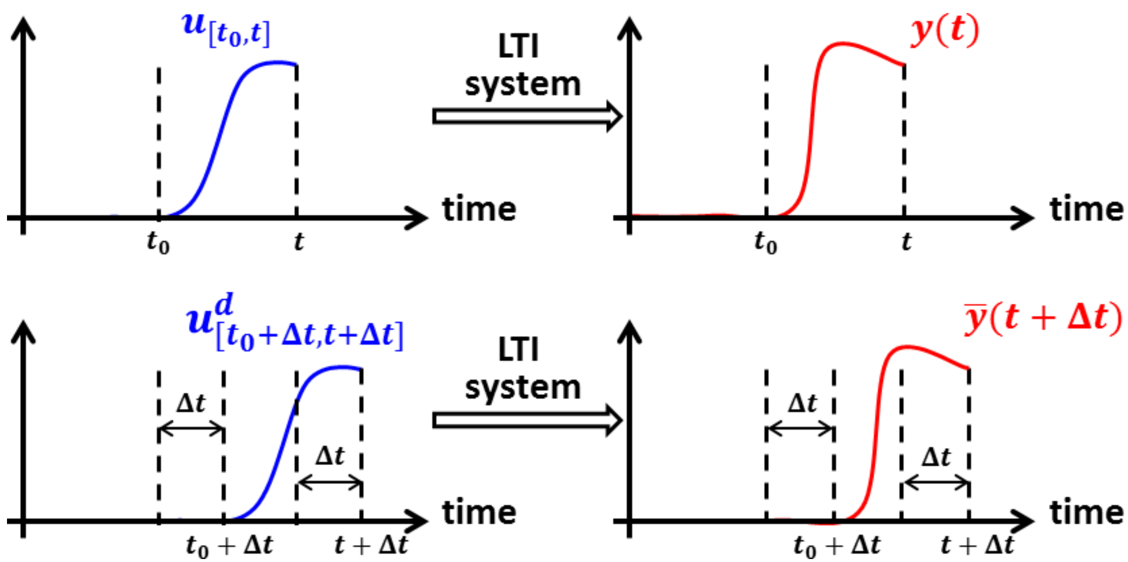
Example 9 Population dynamics

$$N(t+1) = (1+\lambda)N(t)$$

Is this a linear difference equation? Yes! Thus, the system is linear.

Time Invariance

Graphical interpretation:



In words: the response of a time-invariant system to the time-shifted input $u^d(t) \coloneqq u(t - \Delta t)$ is equal to the time-shifted response of the system to the input $u(t)$ (provided initial conditions are the same $x(t_0) = x(t_0 + \Delta t)$).

Put differently: $\bar{y}(t + \Delta t) = y^d(t + \Delta t) = y(t)$.

Let us write $\bar{y}(t+\Delta t)$ explicitly:

$$\bar{y}(t+\Delta t) = L\left[t+\Delta t; t_0+\Delta t, \underbrace{x(t_0+\Delta t)}_{x(t_0)}, u^d_{[t_0+\Delta t, t+\Delta t]}\right]$$
$$= L\left[t+\Delta t; t_0+\Delta t, x(t_0), u^d_{[t_0+\Delta t, t+\Delta t]}\right]$$

Therefore, $\bar{y}(t+\Delta t) = y(t)$ implies:

$$L\left[t+\Delta t; t_0+\Delta t, x(t_0), u^d_{[t_0+\Delta t, t+\Delta t]}\right] = \underbrace{L\left[t; t_0, x(t_0), u_{[t_0,t]}\right]}_{y(t)}$$

Definition 2 *A dynamic system* $y(t) = L[t;\ t_0, x(t_0), u_{[t_0,t]}]$ *is time invariant if*

$$L\left[t+\Delta t; t_0+\Delta t, x(t_0), u^d_{[t_0+\Delta t, t+\Delta t]}\right] = \underbrace{L\left[t; t_0, x(t_0), u_{[t_0,t]}\right]}_{y(t)}$$

for all admissible u, t and Δt, where $u^d(t) = u(t-\Delta t)$

Consequence: for LTI systems the choice of t_0 is arbitrary, therefore $t_0 = 0$ is the easiest choice.

More good news: $y(t) = L[t;\ x(0), u_{[0,t]}]$

Example 10 Show that the system: $\dot{y}(t) = u(t)$ is time invariant*

However, testing the time invariance of a system by definition may be cumbersome. Moreover this test requires us to know L.

For us, the practical test of time-invariance will be the equation itself.

Example 11 Input-output model: $J\ddot{\theta}(t) + B\dot{\theta}(t) + K \underbrace{\theta(t)}_{\text{output}} = \underbrace{\tau(t)}_{\text{input}}$

Do the parameters depend explicitly on time? No! Therefore, the system is time invariant. Alternatively, look at the state-space model. Can you write the system in the form:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \quad - \text{state equation} \\ y(t) &= Cx(t) + Du(t) \quad - \text{output equation} \end{aligned}$$

so that the matrices A, B, C, and D do not depend on t explicitly? Yes!

$$A = \begin{bmatrix} 0 & 1 \\ -K/J & -B/J \end{bmatrix}; B = \begin{bmatrix} 0 \\ 1/J \end{bmatrix}; C = [1 \quad 0]; D = 0$$

Therefore, the system is time invariant!

Example 12 Check whether the following system is time invariant?

$$\dot{x}(t) = u(t)$$
$$y(t) = C(t)x(t)$$

We figure out that $A = 0, B = 1, D = 0,$ but $C = C(t)$ (explicit dependence on time), thus this is a time-varying system.

Explicit dependence on time is indeed necessary for non-stationarity.

Here is a simple example that shows $C \neq const.$, yet the system is not time-varying.

Example 13 Check whether the following system is time invariant?

$$\dot{x}(t) = u(t)$$
$$y(t) = C(x)x(t)$$

We figure out that $A = 0, B = 1, D = 0,$ but $C = C(x)$ (no explicit dependence on time), thus this is a time-invariant system.

Keep in mind that $C = C(x)$ and $x = x(t)$, thus C depends on time. This dependence, however, is through the state vector (implicit).

E.g. $C(x) = -x$, yields:

$$\dot{x}(t) = u(t)$$
$$y(t) = -x^2(t)$$

No parameters depend on t explicitly, thus the system is time-invariant, though nonlinear.

Note: the system (*) does not have to be described by ODE(s).

Example 14 The diffusion of oxygen in a living tissue

$$\frac{\partial y(\xi,\eta,\zeta,\mathrm{t})}{\partial t} = D\left(\frac{\partial^2 y(\xi,\eta,\zeta,t)}{\partial \xi^2} + \frac{\partial^2 y(\xi,\eta,\zeta,t)}{\partial \eta^2} + \frac{\partial^2 y(\xi,\eta,\zeta,t)}{\partial \zeta^2}\right) - ky(\xi,\eta,\zeta,t)$$

Are the parameters of this PDE explicitly time dependent? No! Therefore, the system is time-invariant.

Example 15 Population dynamics

$$N(t+1) = [1+\lambda(t)]N(t)$$

Are the parameters of this difference equation explicitly time dependent? Yes! Thus, the system is time-varying.