Order of a Dynamic System

Brief Recap:

Last time we talked about input-output and state-space models of dynamic systems.

Although we introduced these models using an example of a linear system (eye movement model), the same can be done for nonlinear systems.

$$J\ddot{\theta}(t) + B\dot{\theta}(t) + K \underbrace{\theta(t)}_{\text{output}} = \underbrace{\tau(t)}_{\text{input}}$$

example of an input-output model of a linear system

$$J(\dot{\theta},\theta)\ddot{\theta}(t) + B\left(\dot{\theta}\right)\dot{\theta}(t) + K(\theta) \underbrace{\theta(t)}_{\text{output}} = \underbrace{\tau(t)}_{\text{input}}$$

example of an input-output model of a non-linear system

Recall: the order of the system is equals to the order n of the highest derivative (of the output) in the input-output model. Since the canonical state vector

$$x(t) := [y(t) \;\; dy(t)/dt \;\; \ldots \;\; d^{n-1}y(t)/dt^{n-1}] \quad (1)$$

has n components, we conclude that the order of a system also equals the dimension of its state vector.

The choice of the state vector is not unique. As long as inputs do not contain derivatives in the input-output model, we can define $x(t)$ as in (1) above.

The state variables defined by (1) are called <u>phase variables</u>.

If u comes with derivatives, i.e. $\sum_{k=0}^{n} a_k y^{(k)}(t) = \sum_{k=0}^{n} b_k u^{(k)}(t)$ with $a_n = 1$, the state vector can be defined as:

$$\begin{aligned}
x_1(t) &:= y(t) - b_n u(t) \\
x_2(t) &:= \dot{x}_1(t) + a_{n-1} y(t) - b_{n-1} u(t) \\
&\vdots \\
x_{n-1}(t) &:= \dot{x}_{n-2}(t) + a_2 y(t) - b_2 u(t) \\
x_n(t) &:= \dot{x}_{n-1}(t) + a_1 y(t) - b_1 u(t)
\end{aligned}$$

While we introduced state-space models using an example of a linear system, the same can be done for a nonlinear system.

If a system is linear time invariant, its state-space model can be written in the following form:

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) \qquad &&- \textit{state equation} \\
y(t) &= Cx(t) + Du(t) \qquad &&- \textit{output equation}
\end{aligned}
$$

If a system is nonlinear time invariant, its state-space model *cannot* be written in the form above. Therefore, a more general form is used:

$$
\begin{aligned}
\dot{x}(t) &= f(x(t), u(t)) \qquad &&- \textit{state equation} \\
y(t) &= g(x(t), u(t)) \qquad &&- \textit{output equation}
\end{aligned}
$$

If a system is linear and is given by its state-space model:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

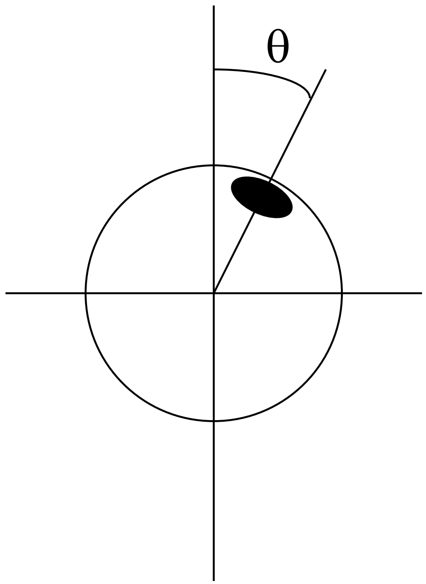
then the order of the system n equals the size of its state-space matrix A. In other words $A \in \mathbb{R}^{n\times n}$

It is easy to why. Since $x(t), \dot{x}(t) \in \mathbb{R}^n$, then:

$$\underbrace{\dot{x}(t)}_{n\times 1} = \underbrace{A}_{n\times n} \underbrace{x(t)}_{n\times 1} + \underbrace{B}_{n\times m} \underbrace{u(t)}_{m\times 1}$$

Also note, m is the number of inputs. If $m > 1$ the system must be MIMO.

The order n of a system has an intuitive meaning. It is the number of independent degrees of freedom that describe the state of the system.



If we know θ and $d\theta/dt$, we know the trajectory of any point on the eyeball (rigid body assumption).

Therefore, the motion of the eyeball is fully described by the knowledge of θ and $d\theta/dt$.

Thus, the number of independent degrees of freedom is 2.

Consequently, this is a second order dynamic system.

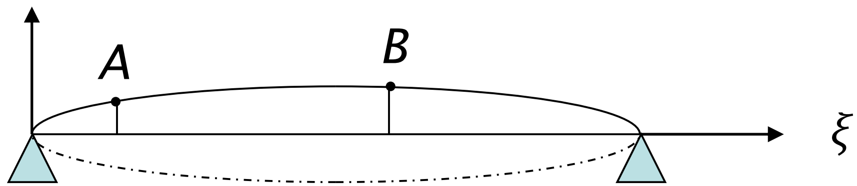
By the same logic, the order of a spatially distributed system is infinity.

In other words, it takes infinitely many degrees of freedom to describe the state of a spatially distributed system.

Example 1 A vibrating string

$$\frac{\partial^2 y(t,\xi)}{\partial t^2} = \theta^2 \frac{\partial^2 y(t,\xi)}{\partial \xi^2}$$

$y(t,\xi)$ – deflection of string



The knowledge of the deflection and velocity of the string at the point A, is not sufficient for knowing the deflection and velocity of the string at the point B.

There are infinitely many such points along the string.

Thus, this is an <u>infinite dimensional system</u>.

What about discrete systems? The logic is just the same.

Example 2 Population dynamics

$$N(t+1) = (1+\lambda)N(t)$$

This is a first order difference equation.

Therefore, the order of the system is $n = 1$.

Example 3 Salmon population (P. H. Wilson, *Conservation Biology*, 2003)

$$S(t) = EpS(t-4) + E(1-p)S(t-5)$$

$S(t)$ – the number of females spawning at year t
E – spawning success
p – the probability of 4 year old spawners
$(1 - p)$ – the probability of 5 year old spawners

For example: $E = 0.57$ and $p = 0.28$ leads to

$$S(t) = 0.16\,S(t-4) + 0.41\,S(t-5)$$

This is a fifth order difference equation.

Therefore, the order of the system is 5.

What would be the state-space model of the system above*?

Canonical Models of Linear Time-Invariant (LTI) Systems

In systems theory there are a couple canonical LTI models that represent a simplified, but sufficiently accurate description, of many systems.

For this reason these models are called canonical models.

<u>Canonical 1st order LTI system</u>: $\boxed{T\,\dot{y}(t) + y(t) = ku(t)}$

T – time constant (tells us how fast the system is)

k – dc gain (tells us how much the system amplifies dc signals (dc refers to signals with zero frequency).

The larger the time constant, the more sluggish the system is, and vice versa.

dc gain gives the ratio of the output and input in the steady state (as $t \to \infty$) when the input $u(t)$ is a signal of a zero-frequency.

By sluggishness we do not mean the absolute speed of the system. We mean how quickly the system reacts to input perturbations.

Large time constants are typically associated with large inertial forces (anything that has a large mass or a moment of inertia) and will make the system more sluggish.

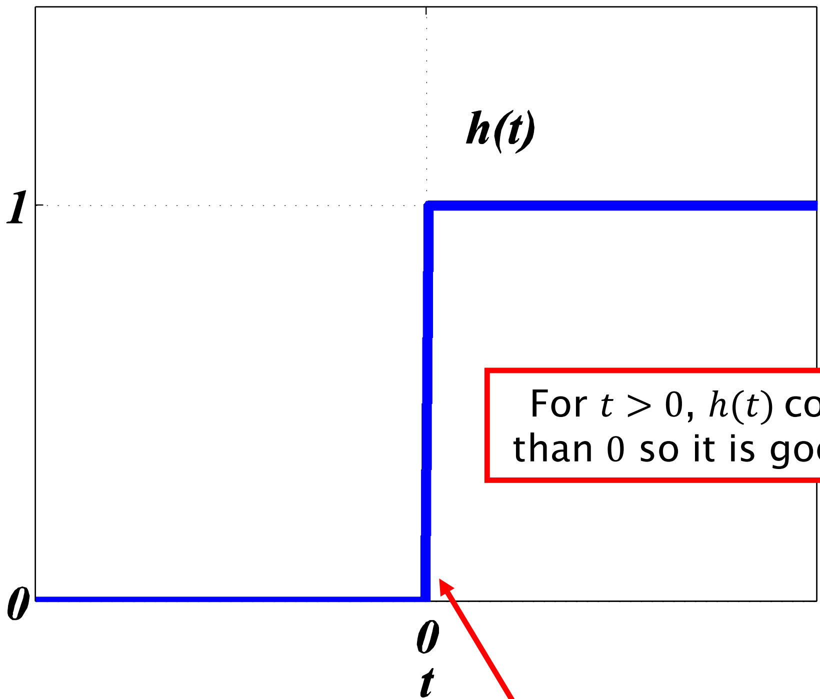
For example: the time constant of a small car is smaller than that of a more massive car, even if the larger can achieve greater speeds.

An input that is ideally suited to characterize both the time constant and the dc gain is the unit step function $h(t)$ (also known as the Heaviside function)

The response of an LTI system to the Heaviside function (under zero initial conditions) is called the unit step response.

$$y(t) = L[t;\, h_{[0,t]}]$$

Heaviside Function

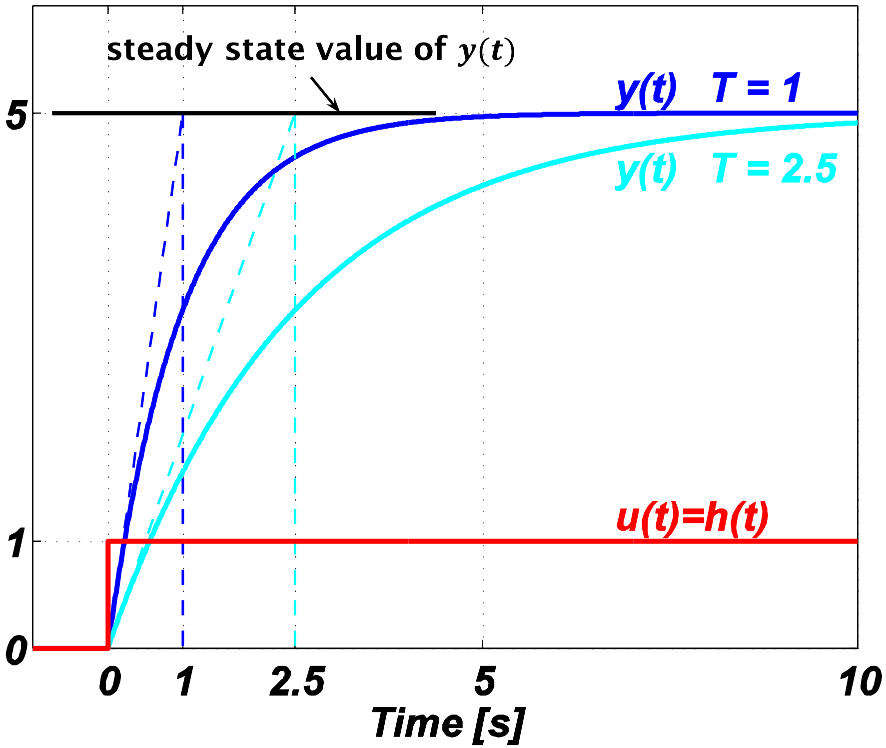


$$h(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

For $t > 0$, $h(t)$ contains no frequencies other than 0 so it is good for measuring the dc gain

The rapid change at $t = 0$ is good for measuring the time constant

Mathematically: $\text{dc gain} \coloneqq \lim_{t\to\infty} \frac{y(t)}{u(t)}\Big|_{u(t)=h(t)} = \lim_{t\to\infty} \frac{y(t)}{h(t)} = \lim_{t\to\infty} \underbrace{y(t)}_{\text{unit step response}}$



Graphical Interpretation

Time constant (T)—draw a tangent to $y(t)$ at $t = 0$, and mark the point where the tangent crosses the steady state value of $y(t)$

dc gain (k)—calculate the steady state value of the unit step response $y(t)$.

$$\underbrace{y(t)}_{\text{unit step response}} = k\left(1 - e^{-\frac{t}{T}}\right)$$

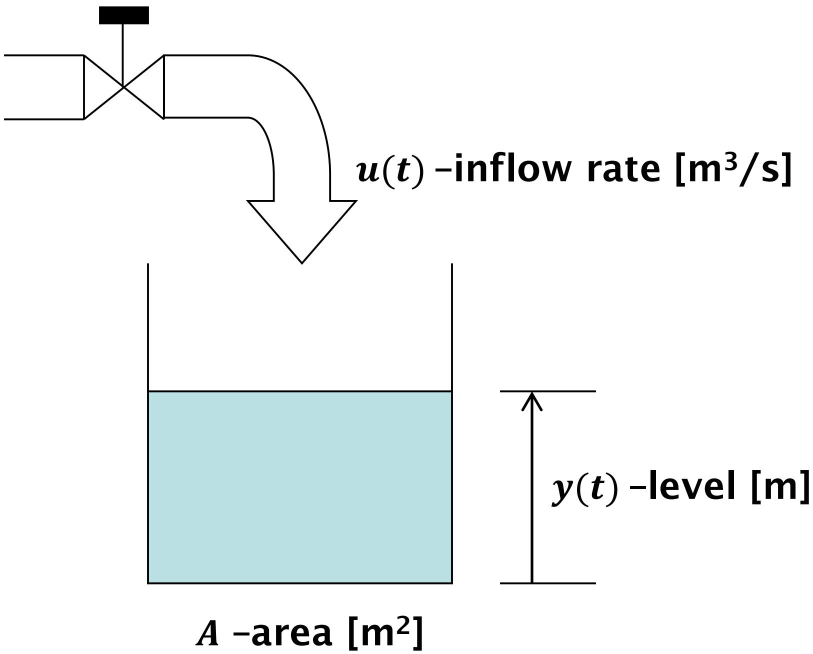
$$\mathbf{1\dot{y}(t) + y(t) = 5u(t)}$$

$$\mathbf{2.5\dot{y}(t) + y(t) = 5u(t)}$$

Demo: `time_constant.m`

The canonical first order LTI system acts like an <u>integrator</u>.

This type of system is sometimes called leaky integrator. Here is why.



$$\frac{dV(t)}{dt} = \text{inflow rate} - \underbrace{\text{outflow rate}}_{0}$$

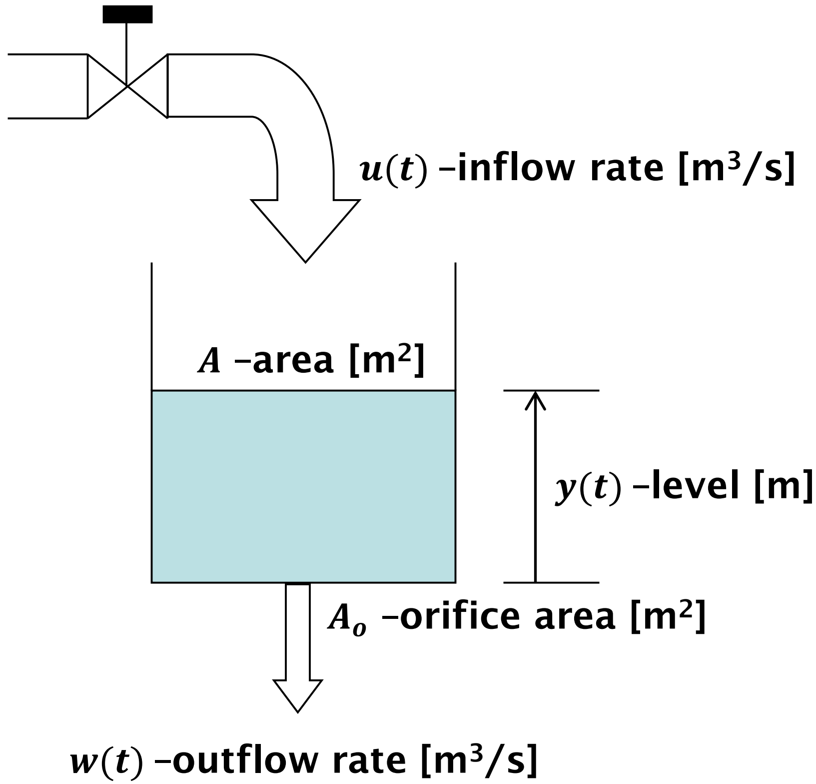
$$\frac{d[Ay(t)]}{dt} = u(t)$$

$$A\,\dot{y}(t) = u(t)$$

If the reservoir is empty initially:

$$y(t) = \frac{1}{A}\int_0^t u(\tau)d\tau$$

Hence the name—the integrator. Note that A plays the role of the time constant. The larger the A the slower the level rises.



$$\frac{dV(t)}{dt} = \text{inflow rate} - \text{outflow rate}$$

$$\frac{d[Ay(t)]}{dt} = u(t) - w(t)$$

$$A\,\dot{y}(t) = u(t) - \underbrace{w(t)}_{\mu A_o\sqrt{2gy(t)}}$$

which follows from the Torricelli Theorem (μ is the discharge coefficient)

$$A\dot{y}(t) + \mu A_o\sqrt{2gy(t)} = u(t)$$

This is a nonlinear input-output model. Linearize the nonlinearity around some nominal level y_N:

$$\sqrt{y(t)} \approx \sqrt{y_N} + \frac{1}{2\sqrt{y_N}}[y(t) - y_N]$$

$$A\dot{y}(t) + \mu A_o\sqrt{2g}\left[\sqrt{y_N} + \frac{1}{2\sqrt{y_N}}[y(t) - y_N]\right] = u(t)$$

$$A\dot{y}(t) + \underbrace{\mu A_o\sqrt{2gy_N}}_{q_N}\left[1 + \frac{y(t) - y_N}{2y_N}\right] = u(t)$$

$$A\dot{y}(t) + \frac{q_N}{2}\frac{y(t) - y_N}{y_N} = u(t) - q_N$$

$$\frac{2A}{q_N}\dot{y}(t) + \frac{y(t) - y_N}{y_N} = 2\,\frac{u(t) - q_N}{q_N}$$

Introduce variables: $z(t) \triangleq (y(t) - y_N)/y_N$ and $v(t) \triangleq (u(t) - q_N)/q_N$.
Note that $y_N = const.$, thus $dy(t)/dt = y_N\, dz(t)/dt$

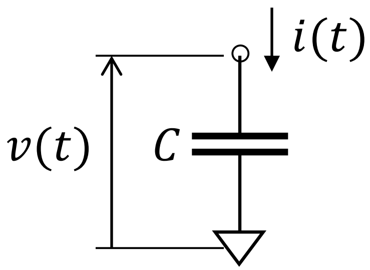
$$\underbrace{\frac{2Ay_N}{q_N}}_{T}\dot{z}(t) + z(t) = \underbrace{2}_{k}\,v(t)$$

Thus, the canonical 1st order LTI system is indeed a leaky integrator.

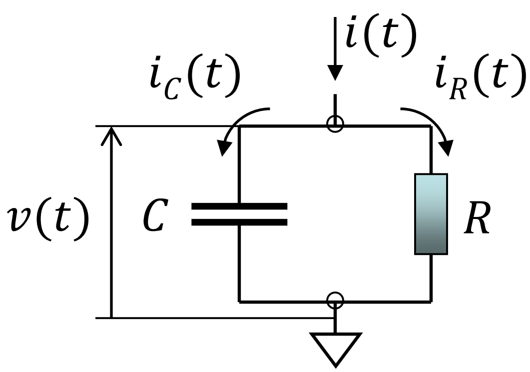
Note that the unit of T is [s], as it should be!

1st order LTI integrators are everywhere:

Example 4 Electric circuits



$$\underbrace{C\dot{v}(t)}_{\text{capacitive current}} = i(t) \quad - \text{integrator}$$



$$i_C(t) + i_R(t) = i(t)$$

$$C\dot{v}(t) + \frac{v(t)}{R} = i(t)$$

$$\underbrace{RC}_{T}\,\dot{v}(t) + v(t) = \underbrace{R}_{k}\,i(t) \quad - \text{leaky integrator}$$

Example 5 Drug delivery dynamics

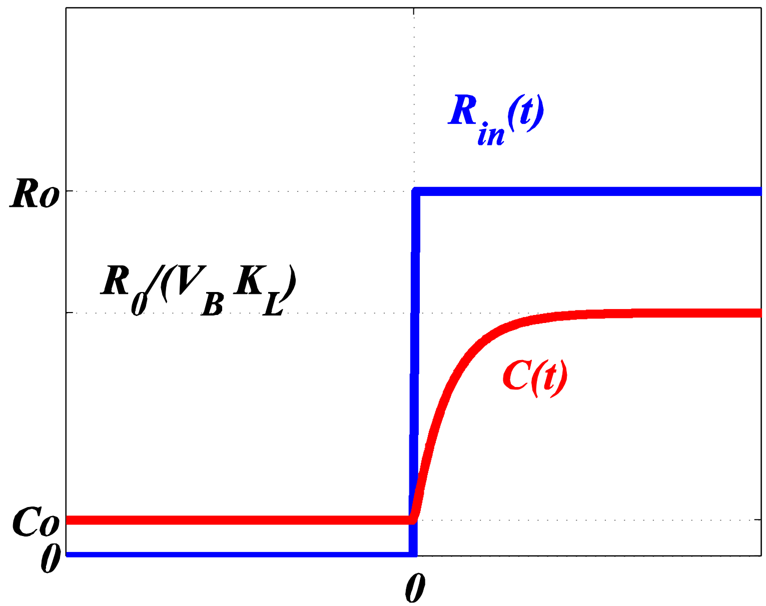
$$\dot{C}(t) = \underbrace{-K_L C(t)}_{\text{liver}} + \underbrace{\frac{1}{V_B} R_{in}(t)}_{\text{I.V. delivery}}$$

C - drug concentration [kg/m^3]
K_L - liver constant [1/s]
V_B - blood volume [m^3]
R_{in} - rate of injection [kg/s]

$$\dot{C}(t) + K_L C(t) = \frac{1}{V_B} R_{in}(t)$$

$$\underbrace{\frac{1}{K_L}}_{T} \dot{C}(t) + C(t) = \underbrace{\frac{1}{V_B K_L}}_{k} R_{in}(t)$$

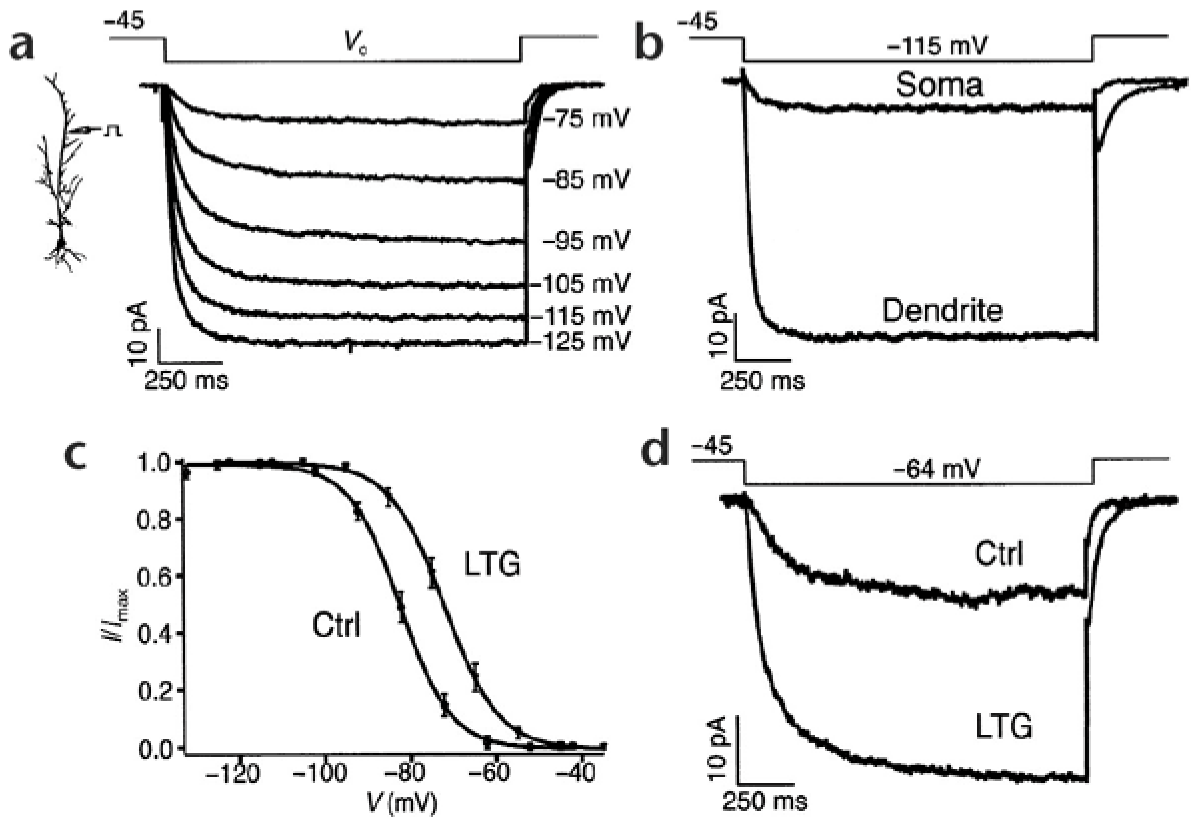
I.V. drip



Example 6

Neural membrane potential

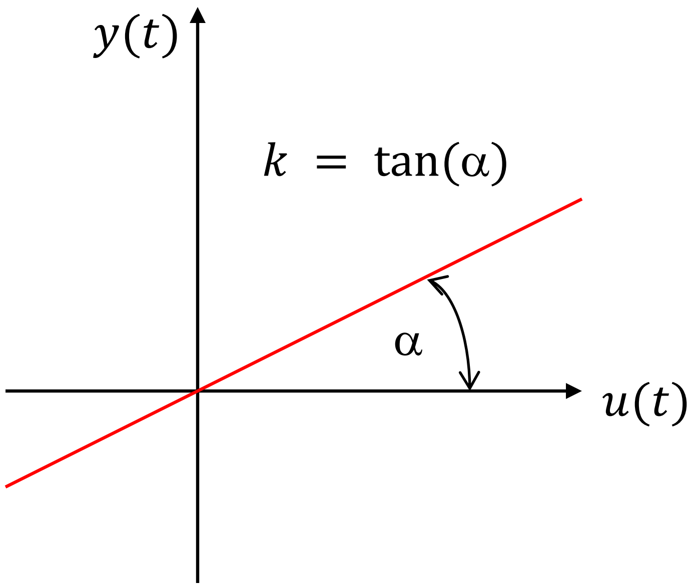
(N.P. Poolos *et al.* *Nature Neurosci*. 2002)



Interesting things happen when $T \to 0$.

$$\underbrace{T}_{\approx 0}\dot{y}(t) + y(t) = ku(t)$$

$$y(t) = ku(t) \quad - \text{static system}$$



Static characteristic (for static linear systems it is a straight line).

We conclude that static LTI systems are infinitely fast ($T = 0$).

That means that they can follow signals of any frequency. In particular they do not perform any filtering.

No real system can follow arbitrarily fast frequencies—static systems do not really exist.

Canonical 2nd order LTI system:

$$\boxed{\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2 y(t) = k\omega_n^2 u(t)}$$

ζ -damping factor ($\zeta \in [0, 1]$)
ω_n -natural frequency
k -dc gain

The new quality of the canonical 2nd order LTI system is oscillatory behavior (something first order systems are not capable of).

ζ tells us how oscillatory the system is:

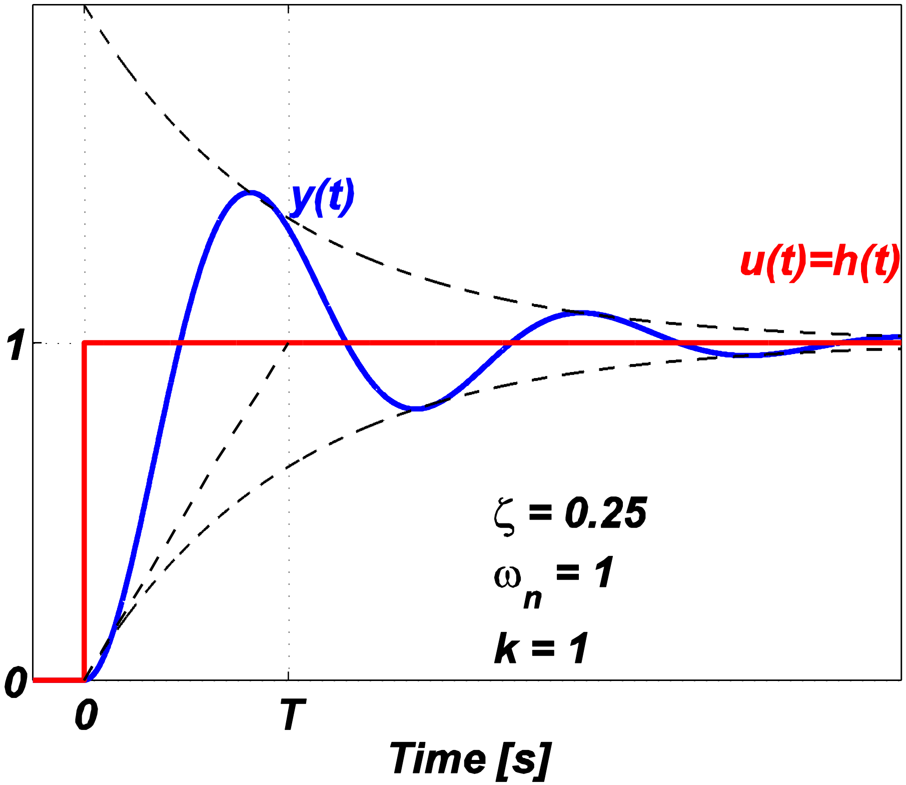
$\zeta = 0$ (undamped system)
$0 < \zeta < 1$ (oscillatory system)
$\zeta = 1$ (critically damped system)
$\zeta > 1$ (overdamped system—not really a canonical 2nd order system)

ω_n tells us which frequency the system is tuned to.

$$\text{dc gain} \coloneqq \lim_{t\to\infty} \frac{y(t)}{u(t)}\Big|_{u(t)=h(t)} = \lim_{t\to\infty} \underbrace{y(t)}_{\text{unit step resp.}}$$

$$\begin{aligned}
\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2 y(t) &= k\omega_n^2 h(t) \\
\lim_{t\to\infty}[\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2 y(t)] &= \lim_{t\to\infty}[k\omega_n^2 h(t)] \\
\underbrace{\lim_{t\to\infty} \ddot{y}(t)}_{0} + 2\zeta\, \omega_n \underbrace{\lim_{t\to\infty} \dot{y}(t)}_{0} + \omega_n^2 \lim_{t\to\infty} y(t) &= k\omega_n^2 \underbrace{\lim_{t\to\infty} h(t)}_{1} \\
\omega_n^2 \lim_{t\to\infty} y(t) &= k\omega_n^2 \\
\text{dc gain} = \lim_{t\to\infty} y(t) &= k
\end{aligned}$$

What about the unit step response of this system $y(t) = L[t;\, h_{[0,t]}]$?



Envelope: $e(t) = k\left(1 \pm e^{-\zeta\omega_n t}\right)$

To find T—draw a tangent to $e(t)$ at $t = 0$, and mark the point where the tangent crosses the steady state value of $e(t)$.

$$T = \frac{1}{\zeta\omega_n}$$

$$\omega = \omega_n\sqrt{1-\zeta^2}$$

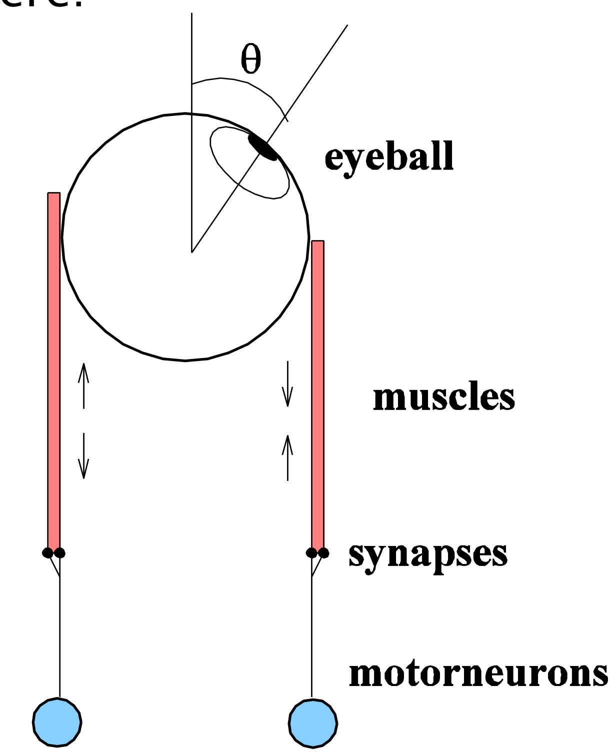
Demo: `second_order_system.m`

If ω_n is large and ζ is not too small, then the system can be simplified:

$$\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2 y(t) = k\omega_n^2 u(t)$$
$$\underbrace{\frac{1}{\omega_n^2}}_{\approx 0}\ddot{y}(t) + \underbrace{\frac{2\zeta}{\omega_n}}_{T}\dot{y}(t) + y(t) = ku(t)$$

Canonical 2nd order LTI systems are everywhere:

Example 7 Eye movement model



$$J\ddot{\theta}(t) + B\dot{\theta}(t) + K\theta = \tau(t)$$

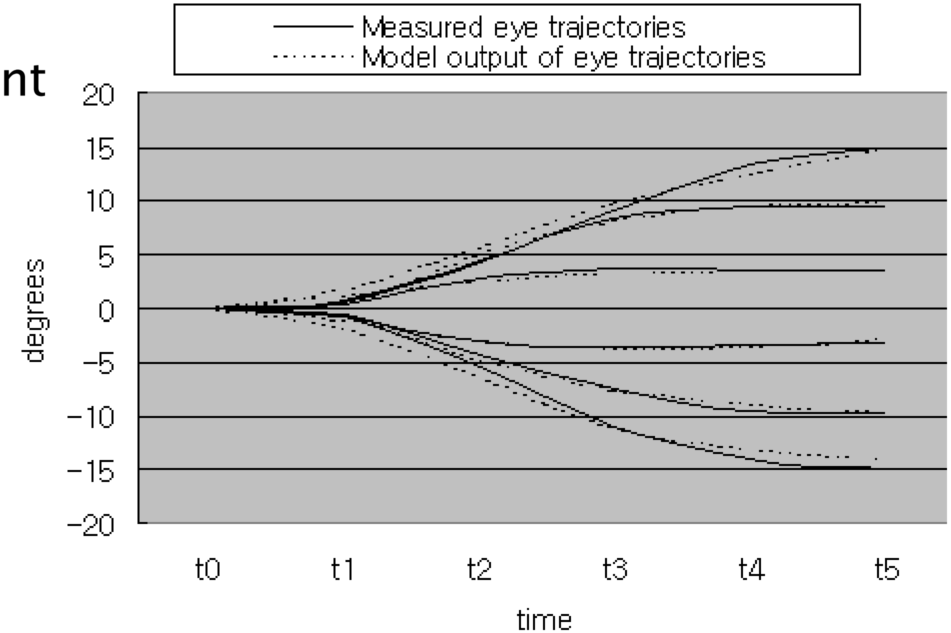
$$\omega_n := \sqrt{\frac{K}{J}}$$

$$\text{if } B^2 - 4KJ < 0, \qquad \zeta := \frac{B}{2\sqrt{KJ}}$$

$$T = \frac{2J}{B}$$

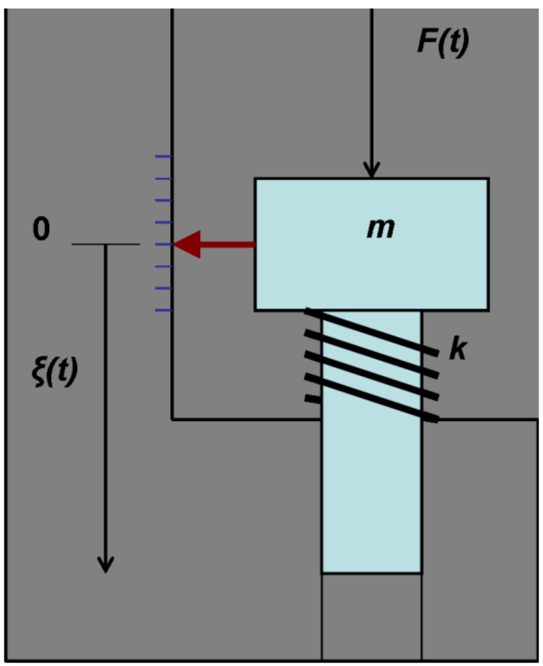
Example 8

Eye movement



Example 9

Spring-mass system (scale)



$$m\ddot{\xi}(t) + b\dot{\xi}(t) + k\xi(t) = F(t)$$

$$\omega_n := \sqrt{\frac{k}{m}}$$

$$\text{if } b^2 - 4km < 0 \quad \zeta := \frac{b}{2\sqrt{km}}$$

$$T = \frac{2m}{b}$$

Example 10

Love affairs (S. Strogatz, *Mathematics Magazine*, 1988)

$$\dot{R}(t) = -aJ(t)$$
$$\dot{J}(t) = bR(t)$$

$R(t)$ – Romeo's love $(R > 0)$/hate $(R < 0)$ for Juliet
$J(t)$ – Juliet's love $(J > 0)$/hate $(J < 0)$ for Romeo
$a, b > 0$

$$\ddot{R}(t) = -a\dot{J}(t) = -abR(t)$$
$$\ddot{R}(t) + abR(t) = 0$$
$$\omega_n = \sqrt{ab};\ \zeta = 0;\ u(t) = 0$$