Impulse Response

To fully characterize an LTI system, it suffices to excite the system using a very special input and observe the system's response.

This special input is called the (unit) impulse function, $\delta(t)$, and the corresponding response (under zero initial conditions) is called the (unit) impulse response.

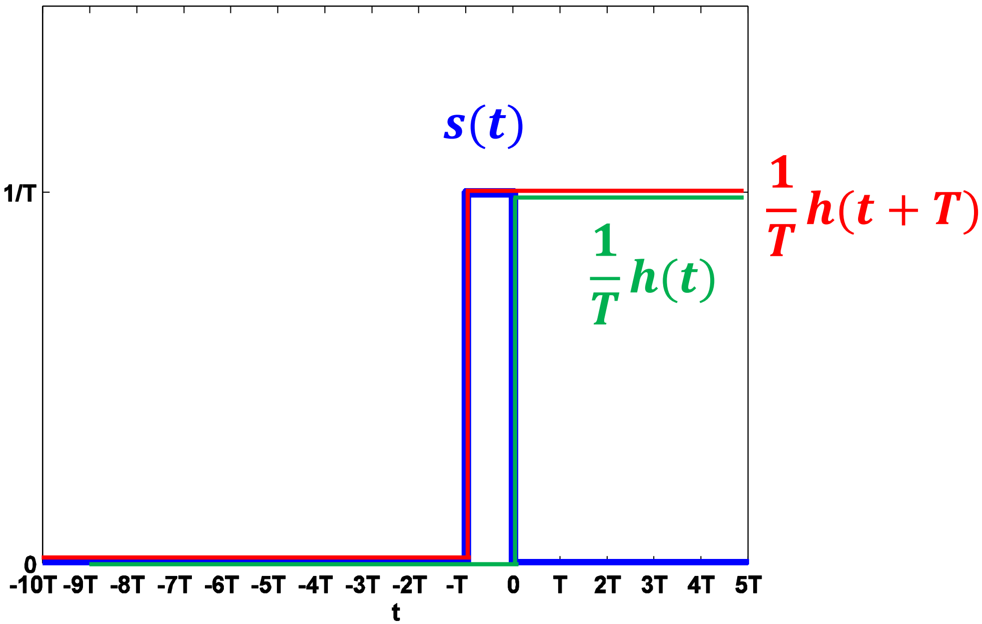
$$y(t) = L[t;\ \delta_{[0,t]}]$$

This response is very special; if we know it, we know everything there is to know about an LTI system. Therefore, the impulse response has its own notation:

$$g(t) = L[t;\ \delta_{[0,t]}]$$

The unit impulse function is also known as the delta function or the Dirac function.

The Dirac function is derived from the unit square pulse function:



$$s(t) \coloneqq \frac{1}{T}[h(t+T) - h(t)]$$

$$\delta(t) = \lim_{T\to 0} s(t) = \lim_{T\to 0}\frac{1}{T}[h(t+T) - h(t)]$$

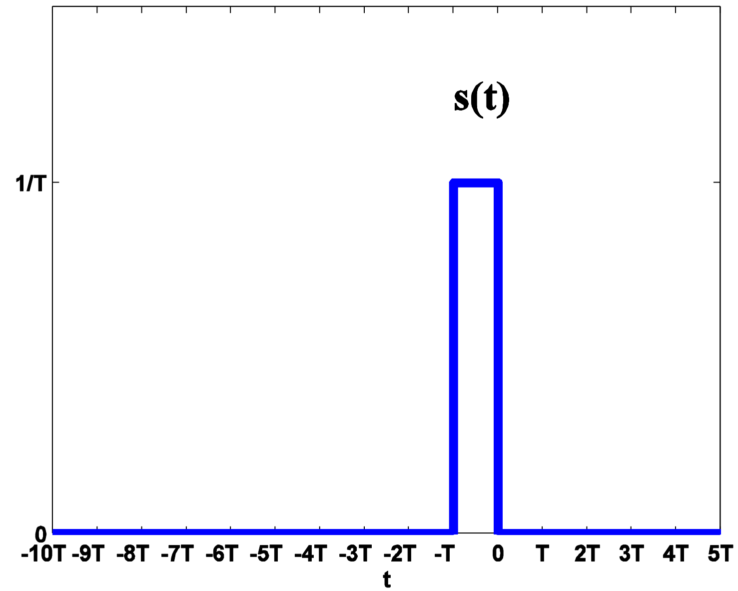
Note: $\int_{-\infty}^{\infty} s(t)dt = \int_{-T}^{0}\frac{1}{T}dt = 1$

Other definitions will also work: $\delta(t) := \lim_{T\to 0}\frac{1}{T}\left[h\left(t+\frac{T}{2}\right) - h\left(t-\frac{T}{2}\right)\right]$, or even

$$\delta(t) := \lim_{T\to 0}\frac{1}{T}[h(t) - h(t-T)]$$

Note: $$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

The area under the Dirac function is 1.



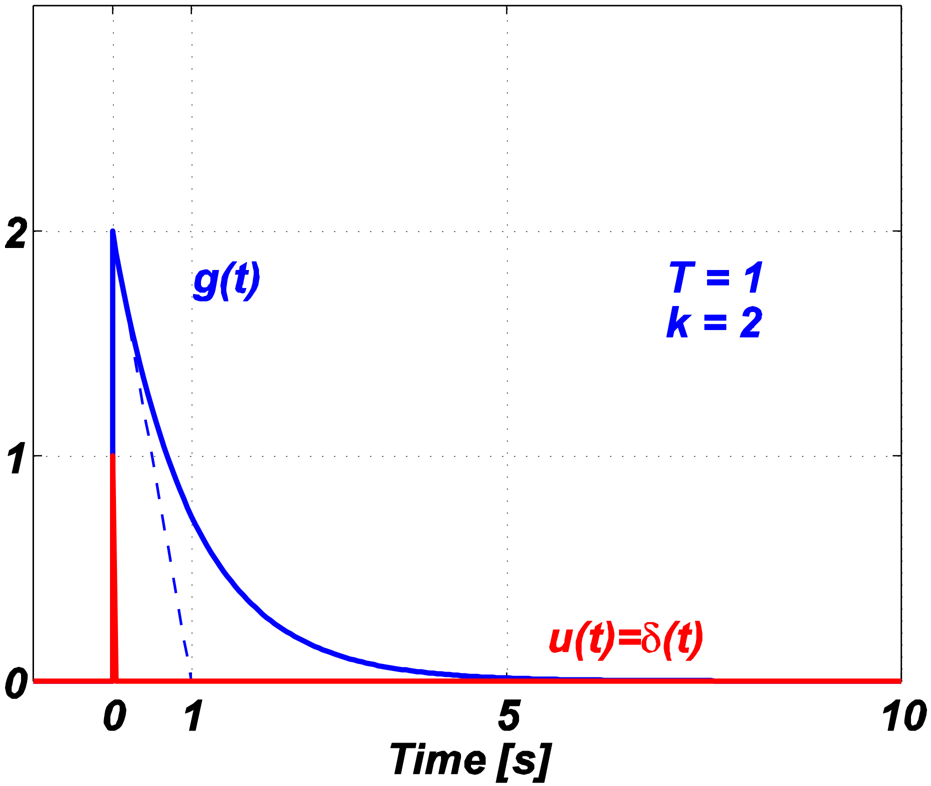
Also note: $$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$$

Proof:

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = \lim_{T\to 0} \int_{-T}^{0} f(t)s(t)dt = f(0) \lim_{T\to 0} \underbrace{\int_{-T}^{0} s(t)dt}_{1} = f(0)$$

Canonical 1st order LTI system:

$$\boxed{T\,\dot{y}(t) + y(t) = ku(t)}$$



Graphical Interpretation

Time constant (T)—draw a tangent to $g(t)$ at $t = 0$, and mark the point where the tangent crosses the steady state value of $g(t)$ which is 0.

dc gain (k)—calculate the initial value of $g(t)$ and set $k = g(0)\ T$.

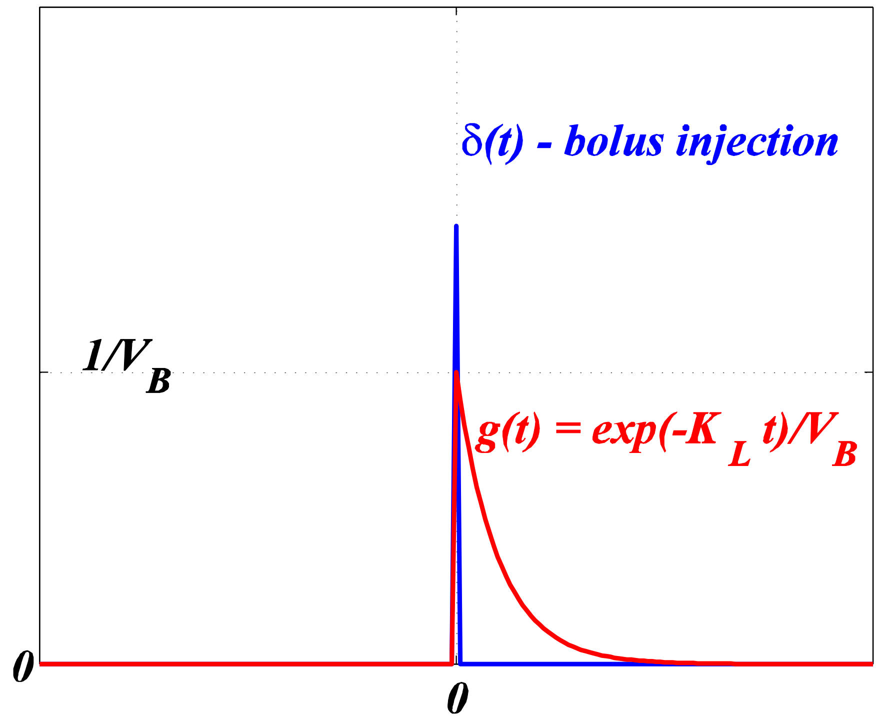
Example 1 Drug delivery dynamics

$$\dot{C}(t) = \underbrace{-K_L C(t)}_{\text{liver}} + \underbrace{\frac{1}{V_B} R_{\text{in}}(t)}_{\text{I.V. delivery}}$$

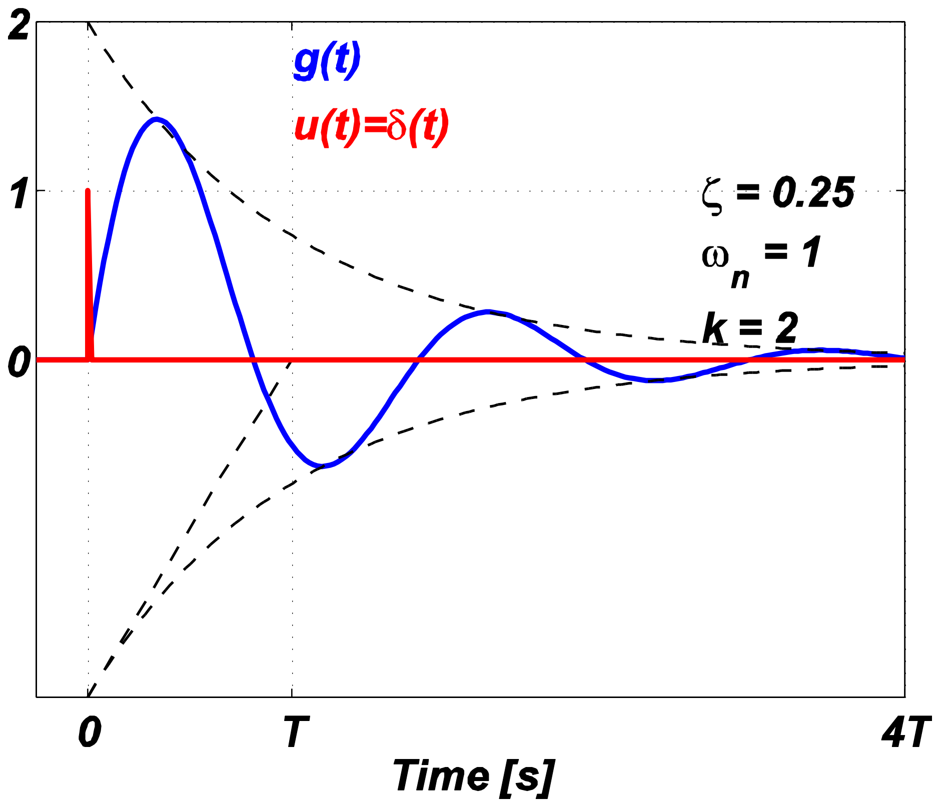
C - drug concentration [kg/m³]
K_L - liver constant [1/s]
V_B - blood volume [m³]
R_{in} - rate of injection [kg/s]

$$\dot{C}(t) + K_L C(t) = \frac{1}{V_B} R_{\text{in}}(t)$$

$$\underbrace{\frac{1}{K_L}}_{T} \dot{C}(t) + C(t) = \underbrace{\frac{1}{V_B K_L}}_{k} R_{\text{in}}(t)$$



Canonical 2nd order LTI system: $\boxed{\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2 y(t) = k\omega_n^2 u(t)}$



Graphical Interpretation

Envelope: $e(t) = ke^{-\zeta\omega_n t}$

To find T—draw a tangent to $e(t)$ at $t = 0$, and mark the point where the tangent crosses the steady state value of $e(t)$ which is 0.

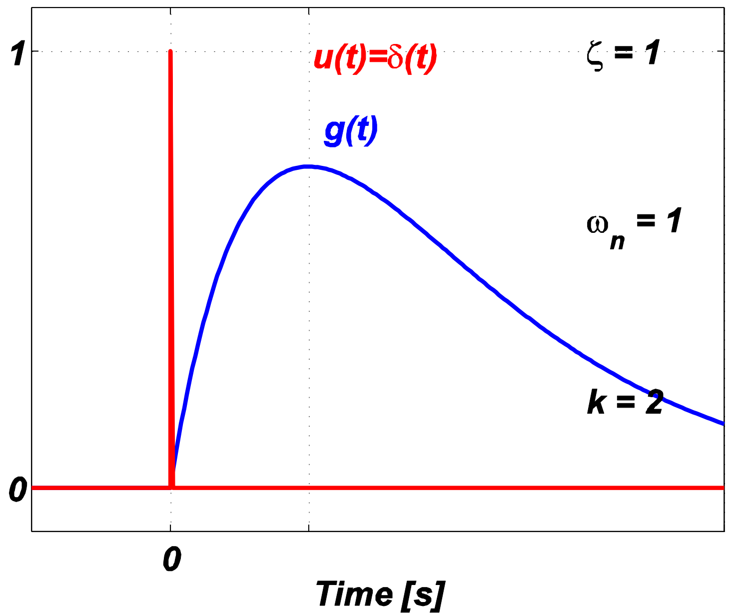
dc gain (k)—calculate the initial value of $e(t)$, that is $k = e(0)$.

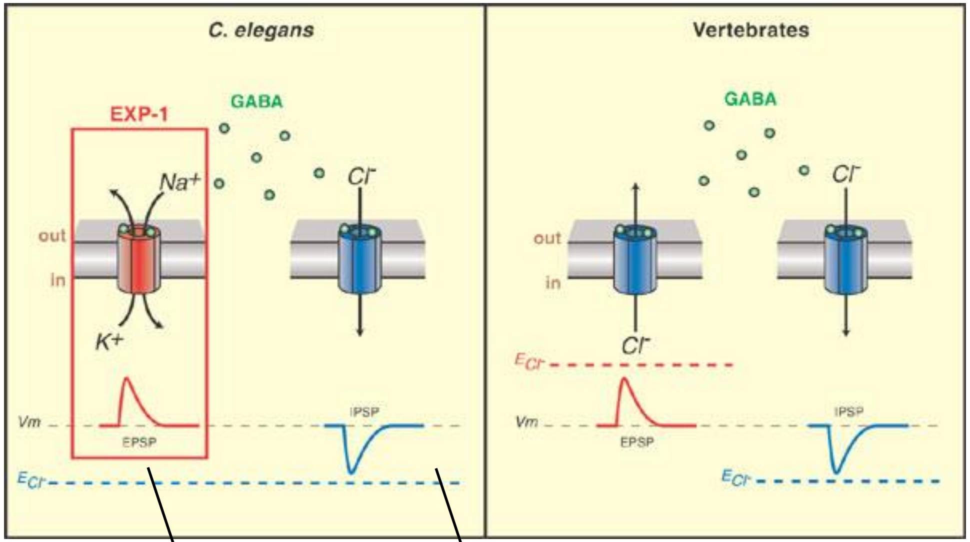
$$T = \frac{1}{\zeta\omega_n}$$

$$\omega = \omega_n\sqrt{1-\zeta^2}$$

Example 2

Excitatory/inhibitory postsynaptic potential (EPSP/IPSP)





Overdamped 2nd order LTI system

excitation

inhibition ($k < 0$)
dc gain negative

Impulse Response of Time-discrete Systems

The special input that fully characterizes a time-discrete LTI system is called the discrete (unit) impulse function, $\delta[k]$, and the corresponding response (under zero initial conditions) is called the discrete (unit) impulse response.

$$y[k] = L[k; \delta_{[0,k]}]$$

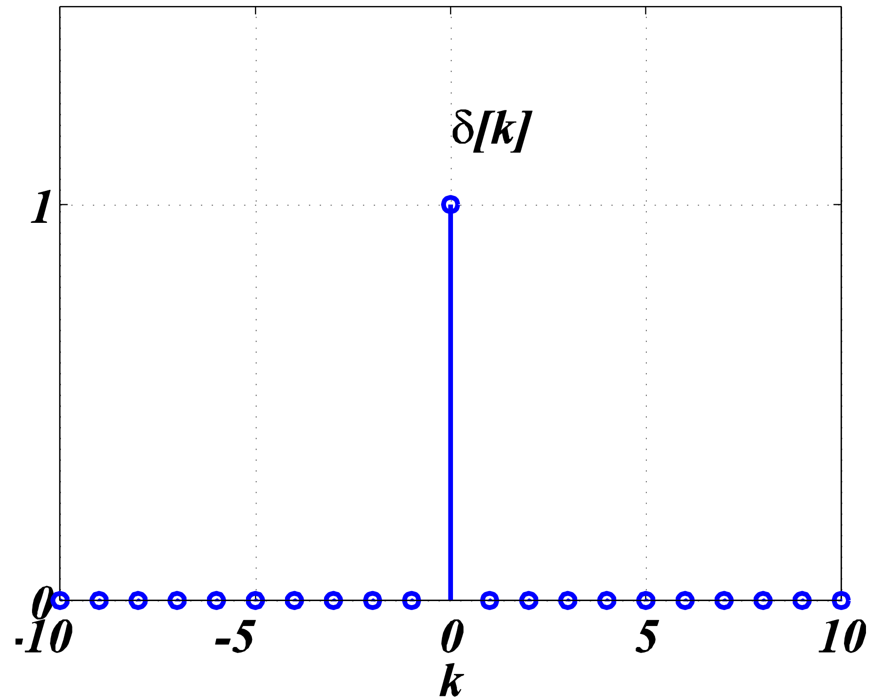
This response is very special, if we know it, we know everything there is to know about the LTI system. Therefore, the impulse response has its own notation:

$$g[k] = L[k; \delta_{[0,k]}]$$

The unit impulse function in the discrete domain is also known as: the delta function or the Kronecker function.

Kronecker function is easily defined

$$\delta[k] = \begin{cases} 1, & k = 0 \\ 0, & \text{ow.} \end{cases}$$



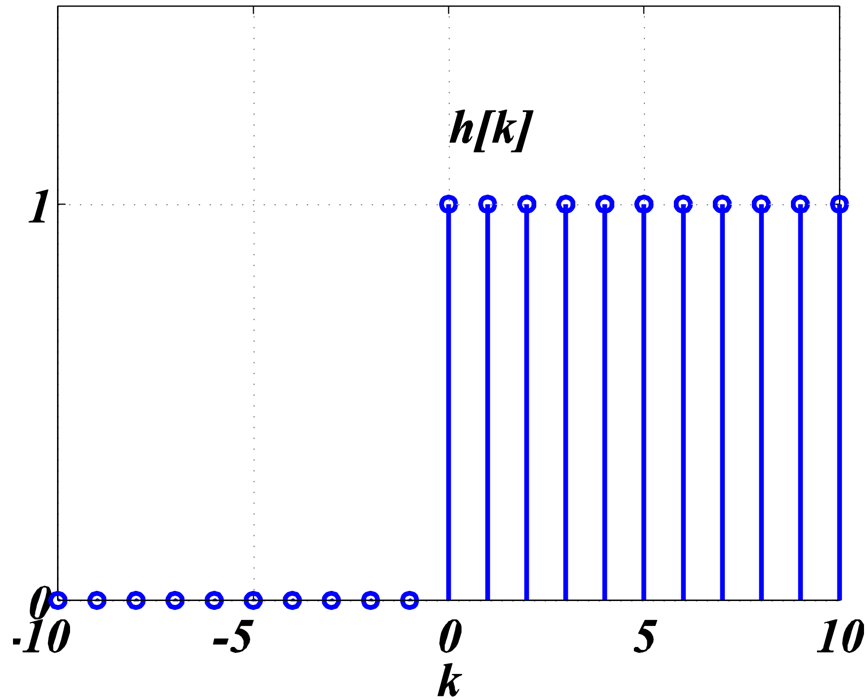
Note that:

$$\sum_{i=-\infty}^{\infty} \delta[i] = 1 \quad \text{and} \quad \sum_{i=-\infty}^{\infty} f[i]\delta[i] = f[0]$$

Also note that: $\delta[k] \coloneqq h[k] - h[k-1]$

where $h[k]$ is the discrete unit step (Heaviside) function.

$$h[k] = \begin{cases} 1, & k = 0,1,\cdots \\ 0, & \text{ow.} \end{cases}$$



The response of a LTI system to the unit step function $h[k]$ (under zero initial conditions) is called the discrete (unit) step response:

$$y[k] = L[k; h_{[0,k]}]$$

Convolution

Recall the state-space model of a time-continuous LTI system.

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \qquad &&-\text{state equation} \\ y(t) &= Cx(t) + Du(t) \qquad &&-\text{output equation} \end{aligned}$$

Let us solve the state equation. The Lagrange (variation of constant) formula:

$$\begin{aligned} \lambda &= A \qquad &&-\text{characteristic equation} \\ x(t) &= e^{At}C(t) \qquad &&-\text{general solution} \\ \dot{x}(t) &= Ae^{At}C(t) + e^{At}\dot{C}(t) && \end{aligned}$$

Note: e^{At} is a matrix. It's called the state transition matrix! Its Maclaurin series expansion is:

$$e^{At} = I + At + \frac{A^2t^2}{2} + \frac{A^3t^3}{6} + \cdots = \sum_{i=0}^{\infty} \frac{A^i t^i}{i!}$$

Plug this back to the state equation:

$$\underbrace{Ae^{At}C(t) + e^{At}\dot{C}(t)}_{\dot{x}(t)} = A\underbrace{e^{At}C(t)}_{x(t)} + Bu(t)$$

$$e^{At}\dot{C}(t) = Bu(t)$$

$$\dot{C}(t) = e^{-At}Bu(t)$$

$$dC(t) = e^{-At}Bu(t)dt$$

$$\int_{t_0}^{t} dC(\tau) = \int_{t_0}^{t} e^{-A\tau}Bu(\tau)d\tau$$

$$C(t) - C(t_0) = \int_{t_0}^{t} e^{-A\tau}Bu(\tau)d\tau$$

$$C(t) = \underbrace{C(t_0)}_{\text{to be found later}} + \int_{t_0}^{t} e^{-A\tau}Bu(\tau)d\tau$$

Go back to the solution $x(t)$:

$$x(t) = e^{At}C(t) = e^{At}\left[C(t_0) + \int_{t_0}^{t} e^{-A\tau}Bu(\tau)d\tau\right]$$

This has to hold for any t, therefore:

$$x(t_0) = e^{At_0}\left[C(t_0) + \underbrace{\int_{t_0}^{t_0} e^{-A\tau}Bu(\tau)d\tau}_{0}\right]$$

$$C(t_0) = e^{-At_0}x(t_0)$$

Go back to the solution $x(t)$:

$$x(t) = e^{At}\left[e^{-At_0}x(t_0) + \int_{t_0}^{t} e^{-A\tau}Bu(\tau)d\tau\right]$$

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

$$x(t) = \underbrace{e^{A(t-t_0)}}_{\text{state transition matrix}} x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)} Bu(\tau) d\tau$$

1) $x(t)$ is the general solution
2) it is good for any choice of admissible input $u(t)$ (operator theory approach, rather than a classical ODE approach)
3) the state transition matrix satisfies the semigroup property:

$$e^{A(t-t_1)} = e^{A(t-t_0)} e^{A(t_0-t_1)} \quad \forall t_0 \in [t_1, t], \forall t_1 < t$$

which has an important corollary:

$$x(t) = e^{A(t-t_0)} x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)} Bu(\tau) d\tau$$

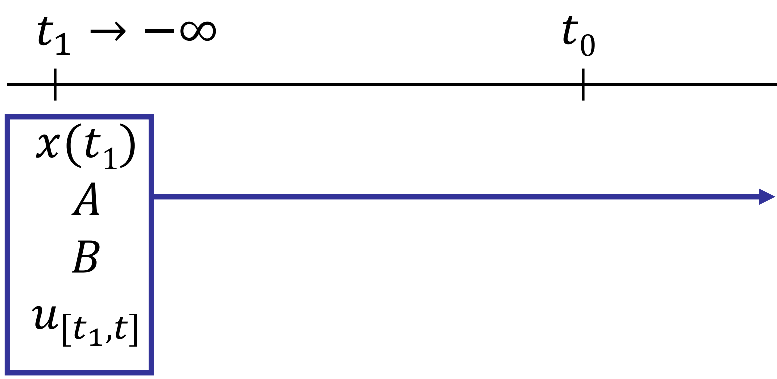
$$x(t_0) = e^{A(t_0-t_1)} x(t_1) + \int_{t_1}^{t_0} e^{A(t_0-\tau)} Bu(\tau) d\tau$$

$$\Downarrow$$

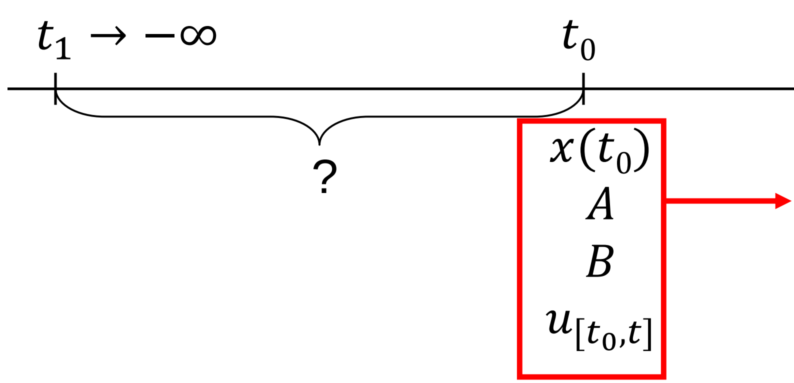
$$x(t) = e^{A(t-t_1)} x(t_1) + \int_{t_1}^{t} e^{A(t-\tau)} Bu(\tau) d\tau$$

Although we start integrating from t_0 onwards, a dynamic system has some history (whatever happened to it prior to t_0).

This history may go very far (in fact to $-\infty$).



$$x(t) = e^{A(t-t_1)}x(t_1) + \int_{t_1}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$



$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

The system's whole history is captured by $x(t_0)$. This is why the concept of state is so important!

$$\boxed{x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau}$$

Going back to the output equation:

$$y(t) = Cx(t) + Du(t)$$
$$y(t) = Ce^{A(t-t_0)}x(t_0) + \underbrace{\int_{t_0}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau}_{\text{convolution}} + Du(t)$$

If the system is causal the output at time t can only depend on inputs prior to time t.

Taken care of by the integral running up to t.

For truly causal systems, even the direct feed-through term ($Du(t)$) is not allowed.

This would mean that $u(t)$ is immediately seen by $y(t)$, i.e. a part of the system would be static.

Thus, for truly causal systems: $D = 0$.

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) && -\text{ state equation}\\ y(t) &= Cx(t) + \underbrace{D}_{0}\,u(t) && -\text{ output equation}\end{aligned}$$

Therefore:

$$y(t) = \underbrace{Ce^{A(t-t_0)}x(t_0)}_{\text{zero-input response (ZIR)}} + \underbrace{\int_{t_0}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau}_{\text{zero-state response (ZSR)}}$$

Recall now the abstract notation: $y(t) = L\left[t; t_0, x(t_0), u_{[t_0,t]}\right]$. We finally know what it is:

$$\begin{aligned}y(t) = L\left[t; t_0, x(t_0), u_{[t_0,t]}\right] &= Ce^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau\\ &= \underbrace{L[t; t_0, x(t_0)]}_{\text{ZIR}} + \underbrace{L\left[t; t_0, u_{[t_0,t]}\right]}_{\text{ZSR}}\end{aligned}$$

Recall that for <u>time-invariant systems</u> the choice of t_0 is irrelevant, thus we'll assume $t_0 = 0$:

$$y(t) = L\big[t; x(0), u_{[0,t]}\big] = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau$$
$$= \underbrace{L[t; x(0)]}_{\text{ZIR}} + \underbrace{L\big[t; u_{[0,t]}\big]}_{\text{ZSR}}$$

Our final claim with respect to convolution: $\underbrace{Ce^{At}B}_{\text{impulse response}} = g(t)$

Impulse response $g(t)$ is the response of the system $y(t)$ when $u(t) = \delta(t)$ and all initial conditions are 0.

$$\underbrace{y(t)}_{g(t)} = L\left[t; \underbrace{x(0)}_{0}, \delta_{[0,t]}\right] = L\big[t; \delta_{[0,t]}\big] = \int_0^t Ce^{A(t-\tau)}B\delta(\tau)d\tau$$
$$= Ce^{At}\underbrace{\int_0^t e^{-A\tau}B\delta(\tau)d\tau}_{B} = Ce^{At}B$$

Therefore: $\boxed{g(t) = Ce^{At}B}$ $-$ impulse response

$$y(t) = L\big[t; x(0), u_{[0,t]}\big] = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau$$

$$= Ce^{At}x(0) + \underbrace{\int_0^t g(t-\tau)u(\tau)d\tau}_{g\star u}$$

Also note (assuming system relaxed at $-\infty$):

$$y(t) = L\big[t; x(-\infty), u_{[-\infty,t]}\big] = Ce^{At}\underbrace{x(-\infty)}_{0} + \int_{-\infty}^t Ce^{A(t-\tau)}Bu(\tau)d\tau$$

$$= \underbrace{\int_{-\infty}^t g(t-\tau)u(\tau)d\tau}_{g\star u} \quad - \text{perhaps a more familiar def. of convolution}$$

Finally, in mathematics: $$f_1 \star f_2 := \int_{-\infty}^{\infty} f_1(t-\tau) f_2(\tau) d\tau$$

Let us assume the system is <u>causal</u> and let us apply the convolution above:

$$\begin{aligned} g \star u &= \int_{-\infty}^{\infty} g(t-\tau)u(\tau)d\tau \\ &= \underbrace{\int_{-\infty}^{t} g(t-\tau)u(\tau)d\tau}_{y(t)} + \underbrace{\int_{t}^{\infty} g(t-\tau)u(\tau)d\tau}_{\text{depends on future inputs } (=0)} \\ &= y(t) \end{aligned}$$

Summary:

$$y(t) = Ce^{At}x(0) + g \star u \quad \text{if} \quad g \star u := \int_{0}^{t} g(t-\tau)u(\tau)d\tau$$

$$y(t) = g \star u \quad \text{if} \quad g \star u := \int_{-\infty}^{t} g(t-\tau)u(\tau)d\tau \quad \text{and} \quad x(-\infty) = 0$$

$$y(t) = g \star u \quad \text{if} \quad g \star u := \int_{-\infty}^{\infty} g(t-\tau)u(\tau)d\tau \quad \text{and} \quad x(-\infty) = 0 \text{ and system causal}$$

For LTI systems, the most common definition is:

$$\boxed{g * u := \int_0^t g(t-\tau)u(\tau)d\tau}$$

So, in summary:

$$\begin{aligned} y(t) &= L\left[t;\, x(0),\, u_{[0,t]}\right] = \underbrace{L\left[t;\, x(0)\right]}_{\text{ZIR}} + \underbrace{L\left[t;\, u_{[0,t]}\right]}_{\text{ZSR}} \\ &= \underbrace{Ce^{At}x(0)}_{\text{ZIR}} + \underbrace{\int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau}_{\text{ZSR}} \\ &= \underbrace{Ce^{At}x(0)}_{\text{ZIR}} + \underbrace{g * u}_{\text{ZSR}} \end{aligned}$$

<u>Take home message</u>: if we know the impulse response $g(t)$ of an LTI system, we know the system's response to any other admissible input $u(t)$.

All we need to do is convolve g and u. This will give us ZSR. Note that ZIR is simple to calculate. Very often we are given $x(0) = 0$, in which case $\text{ZIR} = 0$.

Discrete Convolution

Concept very similar to continuous convolution (a bit easier). Recall the state-space model of a time-discrete LTI system.

$$
\begin{aligned}
x[k+1] &= Ax[k] + B\,u[k] \quad \text{-state equation} \\
y[k] &= Cx[k] + D\,u[k] \quad \text{-output equation}
\end{aligned}
$$

Solution (of the state equation) is straightforward to find:

$$
\begin{aligned}
x[k_0+1] &= Ax[k_0] + Bu[k_0] \\
x[k_0+2] &= A \underbrace{x[k_0+1]}_{Ax[k_0]+Bu[k_0]} + Bu[k_0+1] \\
&= A^2x[k_0] + ABu[k_0] + Bu[k_0+1] \\
x[k_0+3] &= A^3x[k_0] + A^2Bu[k_0] + ABu[k_0+1] + Bu[k_0+2] \\
&\vdots \\
x[k_0+k] &= A^kx[k_0] + A^{k-1}Bu[k_0] + A^{k-2}Bu[k_0+1] + \cdots \\
&+ A^{k-k}Bu[k_0+k-1] = A^kx[k_0] + \sum_{i=k_0}^{k+k_0-1} A^{k+k_0-i-1}Bu[i]
\end{aligned}
$$

$$x[k_0+k] = A^k x[k_0] + \sum_{i=k_0}^{k+k_0-1} A^{k+k_0-i-1} B u[i]$$

substitution: $k + k_0 = j$

$$x[j] = A^{j-k_0} x[k_0] + \sum_{i=k_0}^{j-1} A^{j-i-1} B u[i]$$

Going back to the output equation:

$$y[k] = Cx[k] + D\,u[k]$$

$$y[k] = CA^{k-k_0} x[k_0] + \underbrace{\sum_{i=k_0}^{k-1} CA^{k-i-1} B u[i]}_{\text{convolution}} + D\,u[k]$$

If the system is <u>causal</u> the output at time k can only depend on inputs prior to time k (taken care of by the summation running up to $k-1$).

For truly causal systems, even the direct feed-through ($Du[k]$-term) is not allowed. This would mean that $u[k]$ would be <u>immediately</u> seen by $y[k]$, i.e. the part of the system would be static.

For truly <u>causal</u> systems: $D = 0$.

$$\begin{aligned} x[k+1] &= Ax[k] + B\,u[k] \quad \text{-state equation} \\ y[k] &= Cx[k] + \underbrace{D}_{0}\,u[k] \quad \text{-output equation} \end{aligned}$$

Therefore: $$y[k] = \underbrace{CA^{k-k_0}x[k_0]}_{\text{ZIR}} + \underbrace{\sum_{i=k_0}^{k-1} CA^{k-i-1}Bu[i]}_{\text{ZSR}}$$

Recall the abstract notation: $y[k] = L[k; k_0, x[k_0], u_{[k_0,k]}]$. We finally know what it is:

$$\begin{aligned} L\left[k; k_0, x[k_0], u_{[k_0,k]}\right] &= CA^{k-k_0}x[k_0] + \sum_{i=k_0}^{k-1} CA^{k-i-1}Bu[i] \\ &= \underbrace{L\left[k; k_0, x[k_0]\right]}_{\text{ZIR}} + \underbrace{L\left[k; k_0, u_{[k_0,k]}\right]}_{\text{ZSR}} \end{aligned}$$

For <u>time-invariant systems</u> k_0 is irrelevant (assume $k_0 = 0$):

$$y[k] = L\left[k; x[0], u_{[0,k]}\right] = CA^k x[0] + \sum_{i=0}^{k-1} CA^{k-i-1} B u[i]$$

$$= \underbrace{L\left[k; x[0]\right]}_{\text{ZIR}} + \underbrace{L\left[k; u_{[0,k]}\right]}_{\text{ZSR}}$$

It is trivial to show*: $\underbrace{CA^{k-1}B}_{\text{impulse response}} = g[k]$

$$y[k] = L\left[k; x[0], u_{[0,k]}\right] = CA^k x[0] + \sum_{i=0}^{k-1} CA^{k-i-1} B u[i]$$

$$= \underbrace{CA^k x[0]}_{\text{ZIR}} + \underbrace{\sum_{i=0}^{k-1} g[k-i]u[i]}_{\underbrace{g*u}_{\text{ZSR}}}$$

Discrete convolution: $\boxed{g * u := \sum_{i=0}^{k-1} g[k-i]u[i]}$