The Convolution Theorem

$$\boxed{\mathcal{L}\{g \star u\} = G(s)U(s)}$$

In English: the Laplace Transform of the convolution is the product of the Laplace transforms.

Recall: $$g \star u = \int_0^t g(t-\tau)\, u(\tau)\, d\tau$$

For causal systems ($g(t) = 0$ if $t < 0$): $g \star u = \int_0^\infty g(t-\tau)\, u(\tau)\, d\tau$

$$
\begin{aligned}
\mathcal{L}\{g \star u\} &= \mathcal{L}\left\{\int_0^{\infty} g(t-\tau)u(\tau)\,d\tau\right\} \\
&= \int_0^{\infty}\left[\int_0^{\infty} g(t-\tau)u(\tau)\,d\tau\right]e^{-st}\,dt \\
&= \int_0^{\infty}\underbrace{\left[\int_0^{\infty} g(t-\tau)e^{-st}\,dt\right]}_{\mathcal{L}\{g(t-\tau)\}}u(\tau)\,d\tau \\
&= \int_0^{\infty} e^{-s\tau}G(s)u(\tau)\,d\tau \\
&= G(s)\underbrace{\int_0^{\infty} u(\tau)e^{-s\tau}\,d\tau}_{\mathcal{L}\{u(t)\}} \\
&= G(s)U(s)
\end{aligned}
$$

$\int$ against τ , $\int$ against t.

Transfer Function

Definition 1 *If $g(t)$ is an impulse response of a causal LTI system, then its left Laplace Transform $G^-(s)$ is called the transfer function of the system and is denoted by $T(s)$.*

Theorem 1 *The transfer function $T(s)$ of a causal LTI system is the ratio of the left Laplace transforms of the output $y(t)$ and the input $u(t)$, with all initial conditions set to zero, i.e. $T(s) = Y^-(s)/U^-(s)$.*

Proof (trivial):

$$y(t) = \underbrace{\text{ZIR}}_{0} + \underbrace{\text{ZSR}}_{\text{convolution}} = \int_0^t g(t-\tau)u(\tau)d\tau$$

$$\mathcal{L}^-\{y(t)\} = \mathcal{L}^-\left\{\int_0^t g(t-\tau)u(\tau)d\tau\right\}$$

$$Y^-(s) = G^-(s)U^-(s)$$

$$\underbrace{G^-(s)}_{T(s)\ (\text{Def. } 1)} = \frac{Y^-(s)}{U^-(s)}$$

$$T(s) = \frac{Y^-(s)}{U^-(s)}$$

Note: Some textbooks use the definition $T(s) = Y(s)/U(s)$.

This definition is good in most of the cases. However, one case where $T(s) = Y(s)/U(s)$ would fail is $u(t) = \delta(t)$ (Dirac function). The reason is that $U(s) = \mathcal{L}\{\delta(t)\}$ is not defined.

So why not $\mathcal{L}^+$ then?

$$\begin{aligned} g(t) &= L[t; \delta(t)] \\ \mathcal{L}^+\{g(t)\} &= \mathcal{L}^+\{L[t; \delta(t)]\} \\ G^+(s) &= T(s)\, \underbrace{\mathcal{L}^+\{\delta(t)\}}_{0} = 0 \\ g(t) &= \mathcal{L}^{-1}\{G^+(s)\} = \mathcal{L}^{-1}\{0\} \end{aligned}$$

But we know there should be non-zero response since we excited the system with $\delta(t)$. Try this, however

$$\begin{aligned} g(t) &= L[t; \delta(t)] \\ \mathcal{L}^-\{g(t)\} &= \mathcal{L}^-\{L[t; \delta(t)]\} \\ G^-(s) &= T(s)\, \underbrace{\mathcal{L}^-\{\delta(t)\}}_{1} \\ g(t) &= \mathcal{L}^{-1}\{G^-(s)\} = \mathcal{L}^{-1}\{T(s)\} \end{aligned}$$

<u>Take home message</u>:

$$T(s) = \mathcal{L}^{-}\{g(t)\} = \left[\frac{Y^{-}(s)}{U^{-}(s)}\right]_{\text{ic}=0}$$

$$\boxed{\text{Transfer Function} = \mathcal{L}^{-}\{\text{Impulse Response}\}}$$

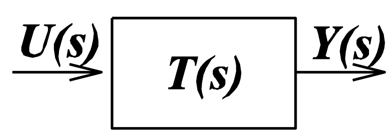
$$\boxed{\text{Transfer Function} = \left[\frac{\mathcal{L}^{-}\{\text{Output}\}}{\mathcal{L}^{-}\{\text{Input}\}}\right]_{\text{ic}=0}}$$

Things to remember:

- Transfer function is a property of LTI systems.
- Transfer function does not depend on the system's input/output. It is inherent to the system.
- Transfer function does not depend on initial conditions, e.g. changing initial conditions will not change the transfer function.

Roughly, it tells us the connection between the input and the output.

Schematically:



Input-output model:

$$\sum_{k=0}^{n} a_k y^{(k)}(t) = \sum_{k=0}^{p} b_k u^{(k)}(t) \qquad a_n = 1$$

$$\mathcal{L}^- \left\{ \sum_{k=0}^{n} a_k y^{(k)}(t) \right\} = \mathcal{L}^- \left\{ \sum_{k=0}^{p} b_k u^{(k)}(t) \right\}$$

$$\sum_{k=0}^{n} a_k s^k Y^-(s) = \sum_{k=0}^{p} b_k s^k U^-(s)$$

$$T(s) = \frac{Y^-(s)}{U^-(s)} = \frac{\sum_{k=0}^{p} b_k s^k}{\sum_{k=0}^{n} a_k s^k}$$

Note: $T(s)$ is a proper rational function $(p < n)$.

$$T(s) = \frac{\sum_{k=0}^{p} b_k s^k}{\sum_{k=0}^{n} a_k s^k} = \frac{r(s)}{q(s)}$$

The polynomial $q(s) = \sum_{k=0}^{n} a_k s^k$ is called the <u>characteristic polynomial</u>.

The zeros of the characteristic polynomial are equal to the roots of the characteristic equation.

Unless $u(t) = \delta(t)$, $\mathcal{L}$ can be used instead of $\mathcal{L}^-$

Example 1 Drug delivery dynamics (canonical 1st order system)

$$\dot{C}(t) = -\underbrace{K_L C(t)}_{\text{liver}} + \underbrace{\frac{1}{V_B} R_{in}(t)}_{\text{I.V. delivery}}$$

C - drug concentration [kg/m^3]
K_L - liver constant [1/s]
V_B - blood volume [m^3]
R_{in} - rate of injection [kg/s]

$$\dot{C}(t) + K_L C(t) = \frac{1}{V_B} R_{in}(t)$$

$$\underbrace{\frac{1}{K_L}}_{\tau} \dot{C}(t) + C(t) = \underbrace{\frac{1}{V_B K_L}}_{k} R_{in}(t)$$

$$\tau \dot{C}(t) + C(t) = k R_{in}(t)$$

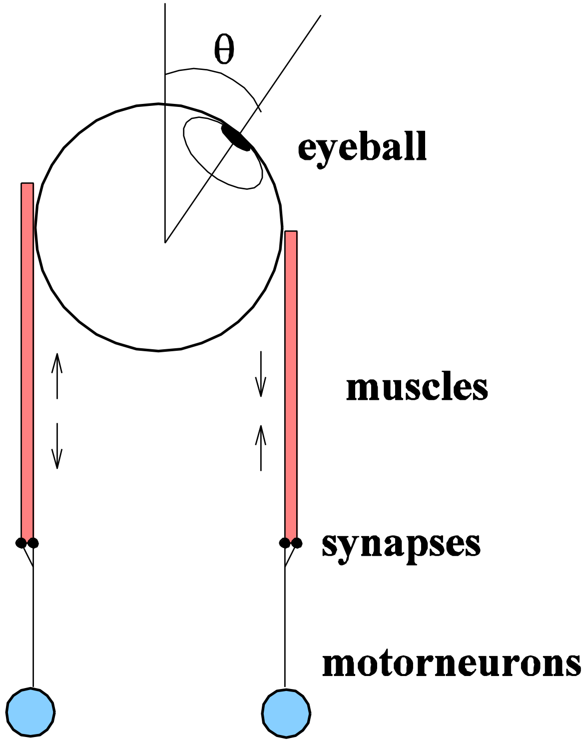
$$\mathcal{L}\left\{\tau \dot{C}(t) + C(t)\right\} = \mathcal{L}\{k R_{in}(t)\}$$

$$(\tau s + 1) C(s) = k R_{in}(s)$$

$$T(s) = \frac{C(s)}{R_{in}(s)} = \frac{k}{\tau s + 1}$$

Example 2 Eye movement model (canonical 2nd order system)

$$J\ddot{\theta}(t) + B\dot{\theta}(t) + K\theta(t) = \tau(t)$$
$$\mathcal{L}\{J\ddot{\theta}(t) + B\dot{\theta}(t) + K\theta(t)\} = \mathcal{L}\{\tau(t)\}$$
$$(Js^2 + Bs + K)\Theta(s) = \tau(s)$$
$$T(s) = \frac{\Theta(s)}{\tau(s)}$$
$$T(s) = \frac{1}{Js^2 + Bs + K}$$



What do we do if we have a state-space model? It's equally easy:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

Take $\mathcal{L}$ of the state equation:

$$\begin{aligned}
\mathcal{L}\{\dot{x}(t)\} &= \mathcal{L}\{Ax(t) + Bu(t)\} \\
sX(s) - \underbrace{x(0)}_{0} &= AX(s) + BU(s) \\
sX(s) - AX(s) &= BU(s) \\
(sI - A)X(s) &= BU(s) \quad (I - \text{identity matrix}) \\
X(s) &= (sI - A)^{-1}BU(s)
\end{aligned}$$

Take $\mathcal{L}$ of the output equation:

$$\begin{aligned}
\mathcal{L}\{y(t)\} &= \mathcal{L}\{Cx(t) + Du(t)\} \\
Y(s) &= CX(s) + DU(s) \\
Y(s) &= C(sI - A)^{-1}BU(s) + DU(s) \\
Y(s) &= \underbrace{[C(sI - A)^{-1}B + D]}_{T(s)}\, U(s)
\end{aligned}$$

Example 3

Canonical first order system:

$$\underbrace{\tau}_{\text{time const.}} \dot{y}(t) + y(t) = \underbrace{k}_{\text{dc gain}} u(t)$$

Immediately find the state-space model as $(x(t) := y(t))$:

$$\dot{x}(t) = -\frac{1}{\tau}x(t) + \frac{k}{\tau}u(t)$$
$$y(t) = x(t)$$

Therefore: $A = -1/\tau$, $B = k/\tau$, $C = 1$, $D = 0$, and

$$T(s) = C(sI - A)^{-1}B + D$$
$$T(s) = 1\left(s \times 1 + \frac{1}{\tau}\right)^{-1}\frac{k}{\tau} + 0$$
$$T(s) = \left(\frac{\tau s + 1}{\tau}\right)^{-1}\frac{k}{\tau} = \left(\frac{\cancel{\tau}}{\tau s + 1}\right)\frac{k}{\cancel{\tau}}$$
$$T(s) = \frac{k}{\tau s + 1}$$

Example 4 Canonical 2nd order LTI system*:

$$\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2 y(t) = k\omega_n^2 u(t)$$

Poles

Definition 2 *Let $T(s) = r(s)/q(s)$ be a transfer function of an LTI system. The zeros of the polynomial $r(s)$ are called the <u>zeros</u> of the system and the zeros of the polynomial $q(s)$ are called the <u>poles</u> of the system.*

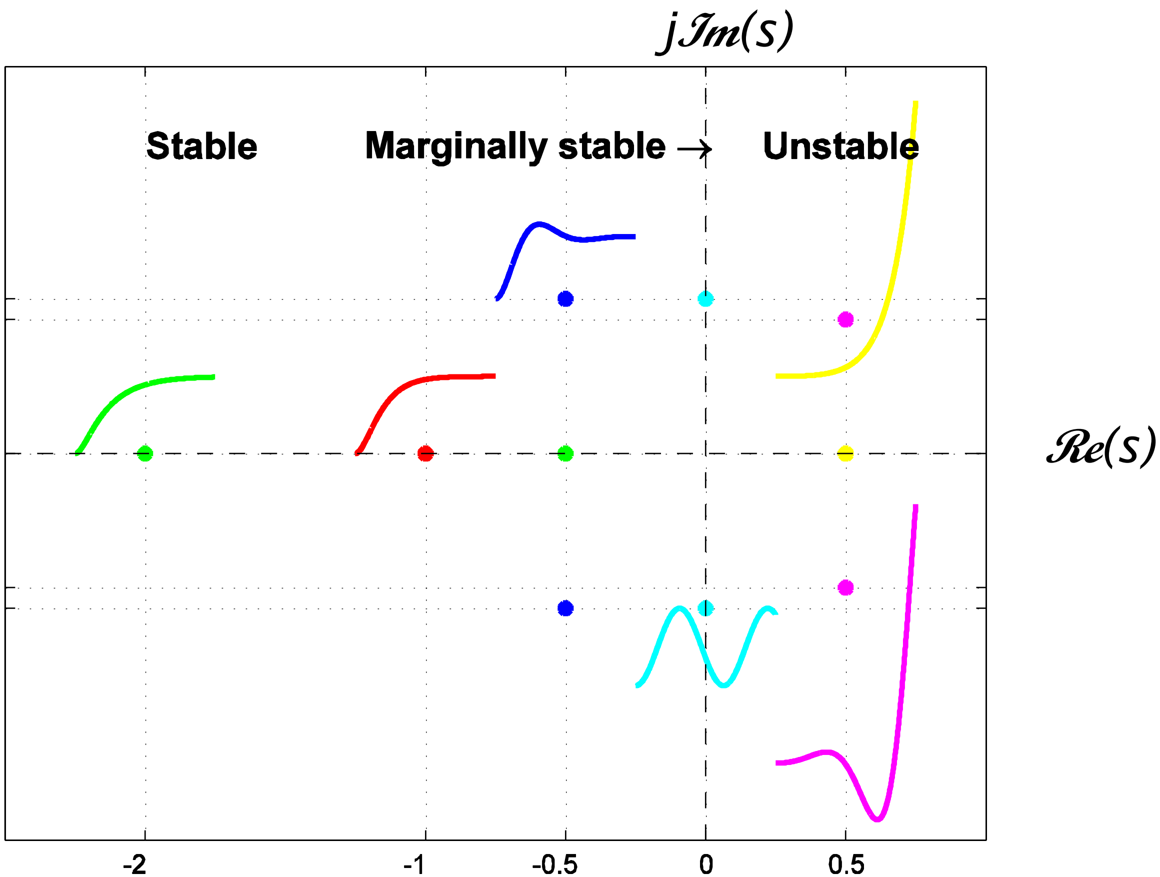
◆ The poles of the system are the same as the roots of the characteristic equation, which are the same as the zeros of the characteristic polynomial, which are the same as the eigenvalues of A (the matrix in the state space model).

Why are they (poles) so great?

The poles tell us whether the system is stable without solving the differential equation. The zeros tell us about the phase shift that the system imposes (more on this later).

Example 5 Unit step response of a 2nd order system

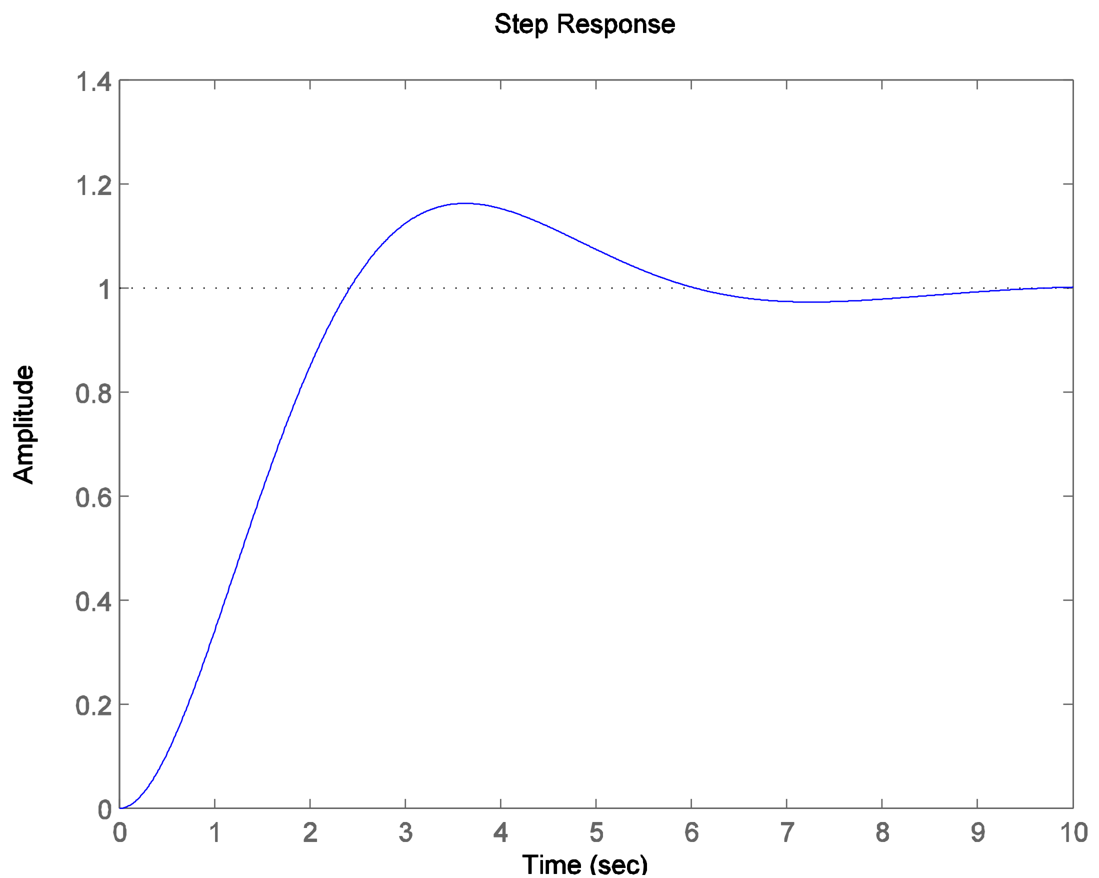
$$T(s) = \frac{b_0}{a_2 s^2 + a_1 s + a_0}$$



The location of the poles determines the step response (in fact any response).

MATLAB function `step.m`

```
>> num = 1; % b0
>> den = [1 1 1]; % a2 a1 a0
>> t = 0:0.01:10; % define time vector
>> step(num,den,t); % get the step response
```



What about the impulse response

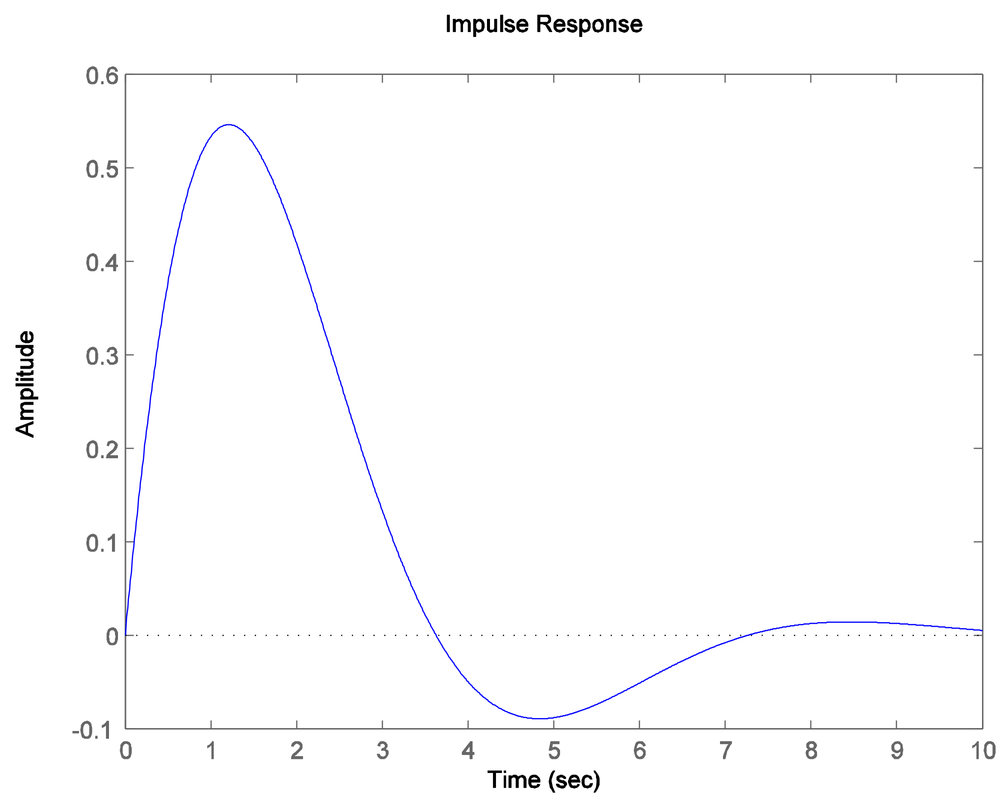
```
>> impulse(num,den,t);
```

will, of course, work. Be adventurous, however:

```
>> sys = tf(num,den)

Transfer function:
1
-----------
s^2 + s + 1

>> impulse(sys,t)
```



The results must be the same.

Equilibrium

Definition 3 *The state of a dynamic system is called the equilibrium state, x_e, if and only if $x(t) = x_e$ for all $t \in [\tau, \infty]$ and under no input conditions.*

In other words, once the system (with 0 inputs) gets into the equilibrium state, it never goes out. Mathematically: if $x(t) = x_e$ for some $t = \tau$, then $x(t) = x_e$ for $t \geq \tau$, unless input is applied.

◆ Note that $dx_e/dt = 0$.

For LTI systems this means:

$$\begin{aligned} \underbrace{\dot{x}_e}_{0} &= Ax_e + B\underbrace{u}_{0} \\ Ax_e &= 0 \\ x_e &= 0 \end{aligned}$$

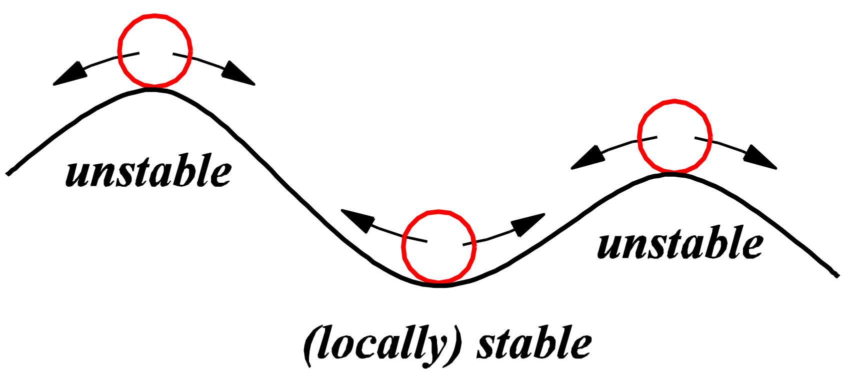
If $\det(A) \neq 0$, it follows from linear algebra that $x_e = 0$ is a <u>unique equilibrium</u> of an LTI system.

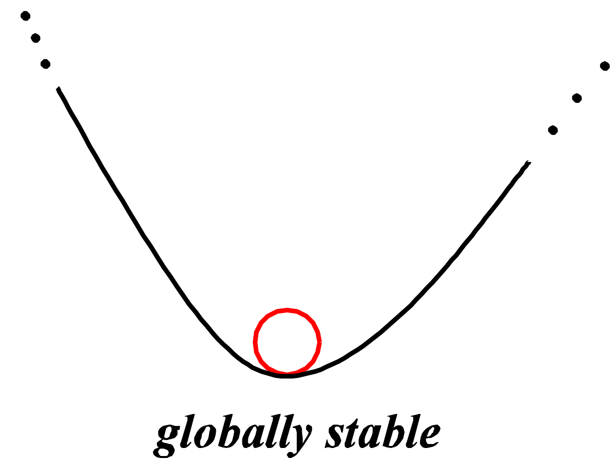
A zero equilibrium, $x_e = 0$, is (locally) stable if the system gets back to x_e under local perturbations.

If locally stable systems are excited only by initial conditions, $x(0)$, the response will eventually die out:

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \mathrm{ZIR} = \lim_{t\to\infty} Ce^{At} \underbrace{x(0)}_{\text{local perturb.}} = 0$$

If this is true for any choice of $x(0)$ we say that the zero equilibrium is globally asymptotically stable.





Stability

Definition 4 *Zero equilibrium is globally asymptotically stable if* $\lim_{t\to\infty} x(t) = 0$ *for any initial condition* $x(0)$.

Definition 5 *An LTI system is stable if its zero equilibrium is globally asymptotically stable.*

Recall: $x(t) = e^{At}x(0)$ (note there are no inputs).
To have $\lim_{t\to\infty} x(t) = 0$ for any choice of $x(0)$, we need $\lim_{t\to\infty} e^{At} = 0$

Theorem 2 *An LTI system is stable if and only if all the eigenvalues of* A *are in left half plane (LHP), i.e.* $Re(\lambda) < 0$, *where* λ *are the eigenvalues of* A.

Intuition: if A is a scalar a, then: $\lim_{t\to\infty} e^{at} = \begin{cases} 0 & a < 0 \\ \infty & a > 0 \\ 1 & a = 0 \end{cases}$

Example 6 Examine the stability of the spring-mass system. Need to find the eigenvalues of A:

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}$$

$$\det(\lambda I - A) = 0$$

$$\det\left(\begin{bmatrix} \lambda & -1 \\ \frac{k}{m} & \lambda + \frac{b}{m} \end{bmatrix}\right) = 0$$

$$\lambda^2 + \frac{b}{m}\lambda + \frac{k}{m} = 0 \quad (b^2 - 4km < 0)$$

$$\lambda_{1,2} = \underbrace{-\frac{b}{2m}}_{\mathcal{R}e(\lambda)} \pm \frac{j}{2}\sqrt{\frac{4k}{m} - \frac{b^2}{m^2}}$$

We know $b > 0$ and $m > 0$, thus the system is stable.

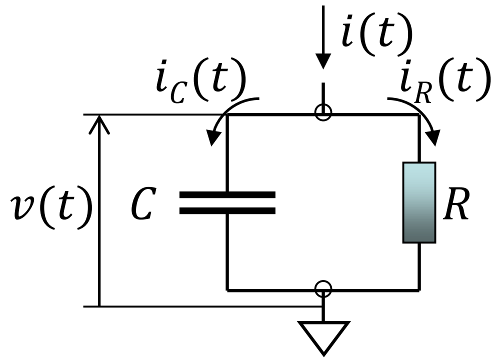
If $b = 0$ (no damping), the system is marginally stable (unstable)

If $b < 0$, the system is unstable.

Play with `spring_mass_eq` demo.

Some systems are inherently stable:

- Spring-mass system (in reality we always have friction)
- RC circuits



- Canonical first order systems
- Canonical second order systems
- and many more

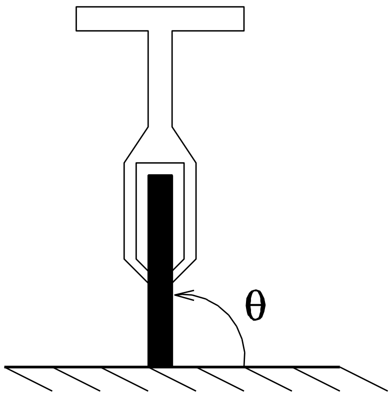
Some systems are inherently unstable:

- nuclear (chain) reaction

- bicycle

3 equilibria:

$\theta = \frac{\pi}{2}, \dot{\theta} = 0$ (unstable),
$\theta = 0, \dot{\theta} = 0$ (stable).
$\theta = \pi, \dot{\theta} = 0$ (stable).



- epileptic seizures

- cancer growth

- diabetes

- etc