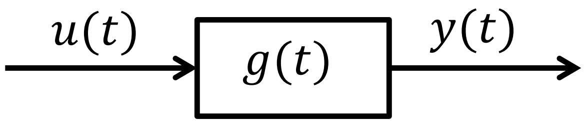
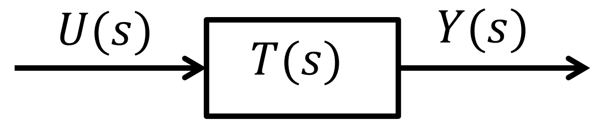
Feedback and Control Systems

Recall that for LTI systems we have:



$$y(t) = (g \star u)(t)$$

(time domain, i.c.=0)

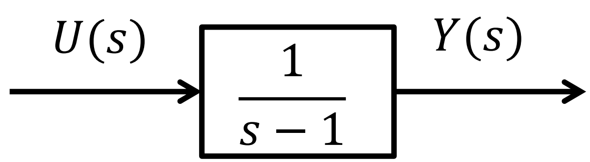


$$Y(s) = T(s)U(s)$$

(complex domain, i.c.=0)

We shall work in the complex domain (a.k.a. frequency domain):

Example:

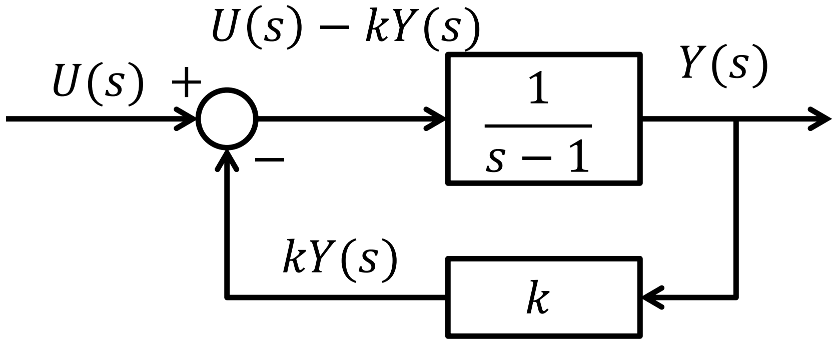


$T(s) = \frac{1}{s-1},\ s^* - 1 = 0 \Rightarrow \underbrace{s^*}_{\text{pole}} = 1\ \ R_e(s^*) > 0$ (unstable system)

If perturbed from the equilibrium $x_e = 0$ the system would never come back to x_e (in fact it would explode)

In particular, if any non-trivial input $u(t)$ is applied, the system's output $y(t)$ would exponentially diverge!

Simple concept—feedback stabilization



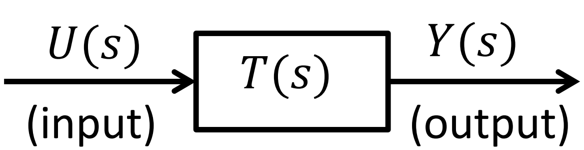
$$Y(s) = \frac{1}{s-1}[U(s) - kY(s)]$$
$$Y(s)(s-1) = U(s) - kY(s)$$
$$Y(s)(s-1+k) = U(s)$$
$$Y(s) = \frac{1}{s-1+k}U(s)$$

The pole: $s^* - 1 + k = 0 \Rightarrow s^* = 1 - k$

For stability we need: $R_e(s^*) < 0$, therefore $1 - k < 0 \Rightarrow k > 1$.

Conclusion: the presence of negative feedback with a sufficiently high gain k stabilizes the system.

The basic idea behind control is simple.

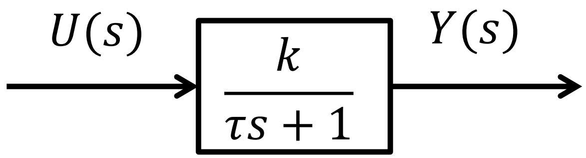


We want: $y(t) \rightarrow \underbrace{y_R(t)}_{\text{reference}}$ (asymptotically)

Goal: find $u(t)$ that will make this possible

Open-loop Control

Example: canonical 1st order system



Assume: $y_R(t) = h(t)$ (Heaviside function)

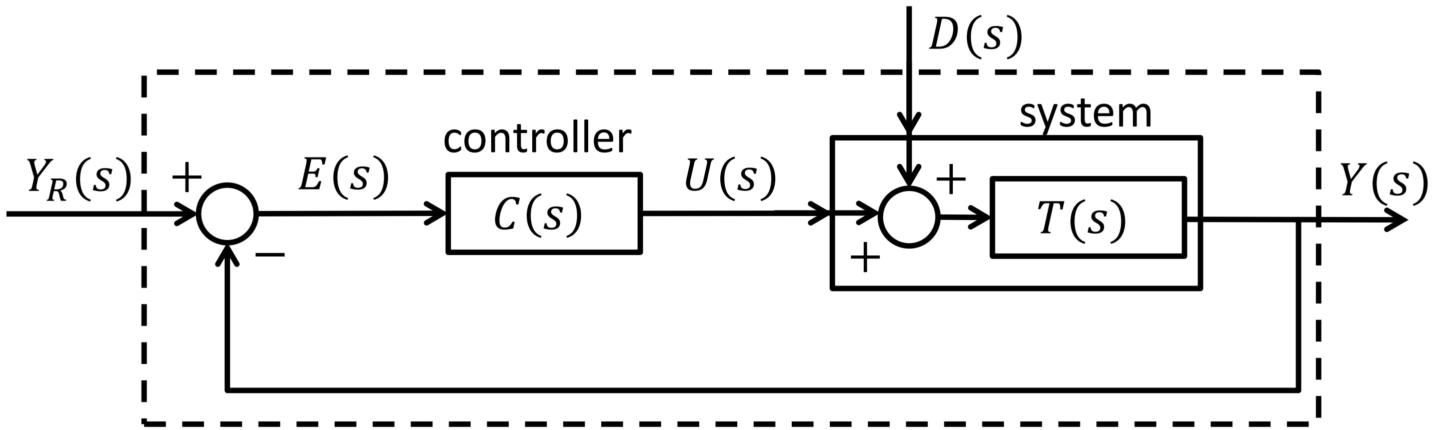
Claim: $u(t) = \frac{1}{k} h(t)$ will achieve $y(t) \to y_R(t)$ as $t \to \infty$

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} sT(s)U(s) = \lim_{s\to 0} s \frac{k}{\tau s + 1} \frac{1}{ks} = \lim_{s\to 0} \frac{1}{\tau s + 1} = \underbrace{1}_{y_R(t)}$$

This control strategy is called <u>open-loop control</u>. It is not very efficient.

- If the parameters k and τ are not accurately estimated, $y(t)$ will diverge from $y_R(t)$
- If the system is affected by a disturbance $d(t)$, $y(t)$ will not follow $y_R(t)$

Feedback Control



Let us calculate the transfer function. Keep in mind that this is a MIMO system, as there are two inputs (Y_R and D) and one output (Y).

$$Y(s) = T(s)[D(s) + U(s)]$$
$$Y(s) = T(s)[D(s) + C(s)E(s)]$$
$$Y(s) = T(s)\big[D(s) + C(s)\big(Y_R(s) - Y(s)\big)\big]$$
$$Y(s) = T(s)D(s) + T(s)C(s)Y_R(s) - T(s)C(s)Y(s)$$
$$Y(s)[1 + T(s)C(s)] = T(s)C(s)Y_R(s) + T(s)D(s)$$
$$Y(s) = \frac{T(s)C(s)}{[1 + T(s)C(s)]}Y_R(s) + \frac{T(s)}{[1 + T(s)C(s)]}D(s)$$

Goal: we want $y(t)$ to track $y_R(t)$ (asymptotically) and we want to reject the disturbance $d(t)$ (asymptotically). Can we do this?

How do we choose $C(s)$ (the controller)?

Example: $T(s) = \frac{k}{\tau s+1}$ 1st order system

$C(s) = k_p$ - proportional controller (static system)

$$Y(s) = \frac{\frac{k}{\tau s + 1} k_p}{\left[1 + \frac{k}{\tau s + 1} k_p\right]} Y_R(s) + \frac{\frac{k}{\tau s + 1}}{\left[1 + \frac{k}{\tau s + 1} k_p\right]} D(s)$$

$$Y(s) = \frac{k k_p}{\tau s + 1 + k k_p} Y_R(s) + \frac{k}{\tau s + 1 + k k_p} D(s)$$

It may be challenging to test the behavior of the closed-loop system for arbitrary Y_R and D. Typically, unit-step changes are assumed, i.e. $Y_R(s) = \frac{1}{s}$ and $D(s) = \frac{1}{s}$

Therefore: $Y(s) = \frac{k k_p}{\tau s+1+k k_p} \frac{1}{s} + \frac{k}{\tau s+1+k k_p} \frac{1}{s}$

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} s\left(\frac{k k_p}{\tau s + 1 + k k_p}\frac{1}{s} + \frac{k}{\tau s + 1 + k k_p}\frac{1}{s}\right)$$

$$= \underbrace{\frac{k k_p}{1 + k k_p}}_{\text{want this to be 1}} + \underbrace{\frac{k}{1 + k k_p}}_{\text{want this to be 0}}$$

If k_p is very large ($k_p \gg k$): $\frac{kk_p}{1+kk_p} \to 1$, and $\frac{k}{1+kk_p} \to 0$

however, the tracking will not be perfect (there will be little bias in $y(t)$).

$C(s) = k_d s$ - differential controller

$$Y(s) = \frac{\frac{k}{\tau s+1}k_d s}{\left[1+\frac{k}{\tau s+1}k_d s\right]}Y_R(s) + \frac{\frac{k}{\tau s+1}}{\left[1+\frac{k}{\tau s+1}k_d s\right]}D(s)$$

$$Y(s) = \frac{kk_d s}{\tau s+1+kk_d s}Y_R(s) + \frac{k}{\tau s+1+kk_d s}D(s)$$

Assume $Y_R(s) = \frac{1}{s}$ and $D(s) = \frac{1}{s}$

Therefore: $Y(s) = \frac{kk_d s}{\tau s+1+kk_d s}\frac{1}{s} + \frac{k}{\tau s+1+kk_d s}\frac{1}{s}$

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} s\left(\frac{kk_d s}{\tau s+1+kk_d s}\frac{1}{s} + \frac{k}{\tau s+1+kk_d s}\frac{1}{s}\right)$$

$$= \underbrace{0}_{\text{wanted this to be 1}} + \underbrace{k}_{\text{wanted this to be 0}}$$

Conclusion: D-controller has no effect on either the tracking of Y_R or rejection of D.

Finally, $C(s) = k_i \frac{1}{s}$ - integral controller

$$Y(s) = \frac{\frac{k}{\tau s + 1} k_i \frac{1}{s}}{\left[1 + \frac{k}{\tau s + 1} k_i \frac{1}{s}\right]} Y_R(s) + \frac{\frac{k}{\tau s + 1}}{\left[1 + \frac{k}{\tau s + 1} k_i \frac{1}{s}\right]} D(s)$$

$$Y(s) = \frac{k k_i}{(\tau s + 1)s + k k_i} Y_R(s) + \frac{k s}{(\tau s + 1)s + k k_i} D(s)$$

Assume $Y_R(s) = \frac{1}{s}$ and $D(s) = \frac{1}{s}$

Therefore: $Y(s) = \frac{k k_i}{(\tau s+1)s+k k_i} \frac{1}{s} + \frac{k s}{(\tau s+1)s+k k_i} \frac{1}{s}$

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} s\left(\frac{k k_i}{(\tau s + 1)s + k k_i} \frac{1}{s} + \frac{k s}{(\tau s + 1)s + k k_i} \frac{1}{s}\right)$$

$$= \underbrace{\frac{k k_i}{k k_i}}_{1} + \underbrace{0}_{\text{as desired}}$$

Conclusion: I-controller achieves exactly what is necessary (asymptotically tracks $y_R(t)$ and rejects $d(t)$). Depending on $T(s)$, a controller is chosen. Combinations of the above controllers are common: $C(s) = k_p + k_d s$ (PD), $C(s) = k_p + k_i \frac{1}{s}$ (PI), $C(s) = k_d s + k_i \frac{1}{s}$ (ID), $C(s) = k_p + k_d s + k_i \frac{1}{s}$ (PID)