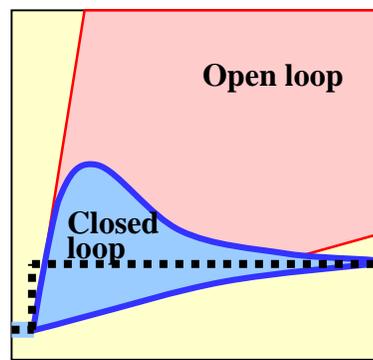
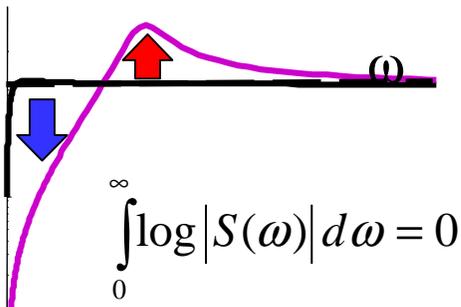
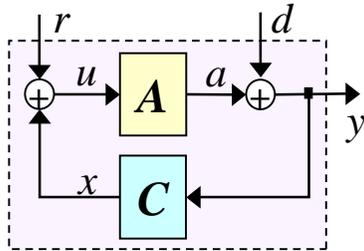
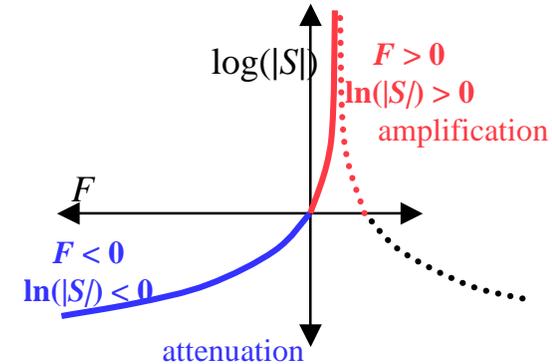


Summary of Elementary Feedback Concepts



$$y(t) \rightarrow \frac{k_2}{k_1} r$$



- This collection of notes expands briefly on the feedback control example studied in the article.
- For additional details on protocols and feedback, proofs of the main results, and examples of biological and engineering applications see www.cds.caltech.edu/~doyle
- Feedback control is both the most powerful and most dangerous protocol for robustness in complex systems.
- The complexity of both engineering and biology are dominated by the sensors, actuators, communications components, and computational elements that implement feedback control.
- This complexity remains hidden as long as it works.
- Integral feedback is both necessary and sufficient for asymptotic steady state tracking of reference r robustly to disturbances d and variations in parameters. It is used ubiquitously in engineering and biology.
- Robust closed loop systems can be built from uncertain components.
- Feedback interconnection has its own “conservation law” for fragility. This and related tradeoffs dominate the design of complex systems.
- This is a mere tip of the iceberg and a rich theory of interconnected systems has been developed within the domain of robust control.

Elementary Feedback Concepts

This set of notes gives additional details on feedback, integral control, and “conservation of fragility.” The simplest possibility is for A and C in Figure 2 to be 1st-order differential equations.

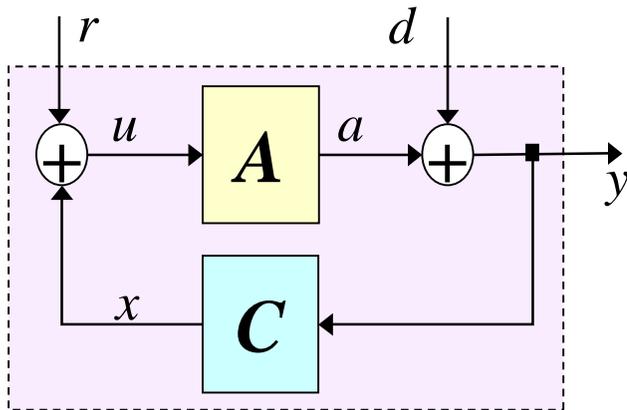


Figure 2. Minimal feedback system with actuator A and controller/sensor C . Goal is for response y to amplify reference r , independent of external disturbance d , and variations in A . The signals u and a are the input and output of the actuator A , and x is the output of C .

$$C: \quad \dot{x} = -k_1 y - k_2 x \quad y = d + a$$

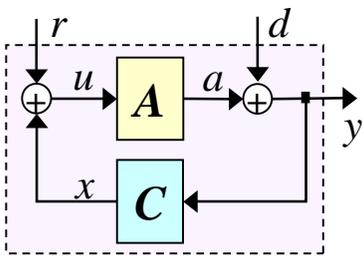
$$A: \quad \dot{a} = gu \quad u = r + x$$

$$\text{Notation: } \dot{x} \triangleq \frac{dx}{dt}$$

C is a low pass filter with internal state x and parameters $k_1 > 0$ and $k_2 > 0$. A is a pure integrator with state a and gain $g > 0$. This doesn't model any particular system, but is a simple, generic feedback example similar to what might arise in a variety of settings in engineering and biology.

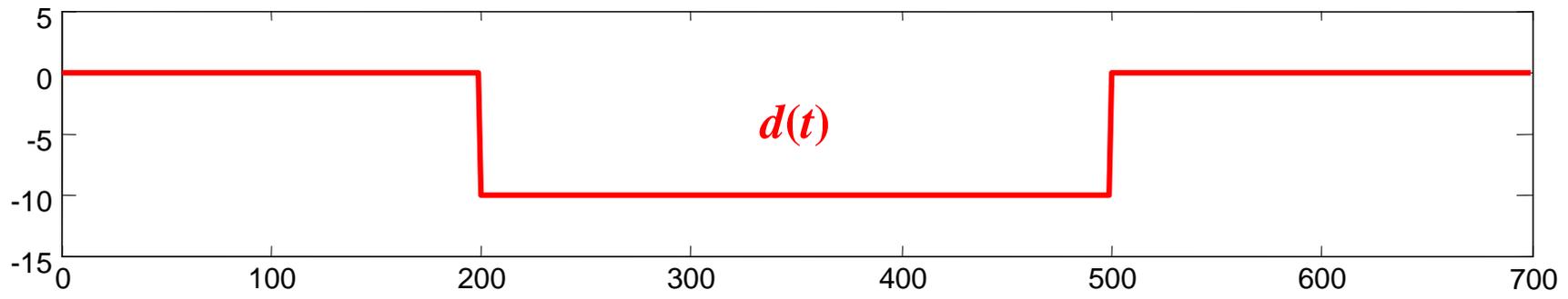
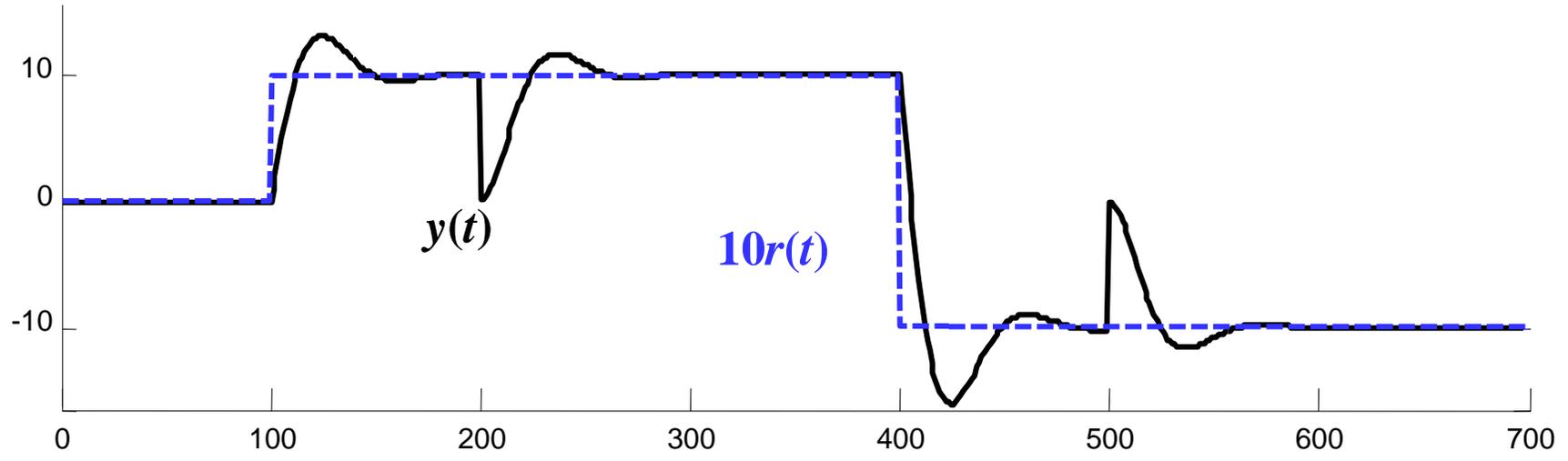
The parameters g , k_1 , and k_2 might typically be functions of underlying physical quantities such as temperature, binding affinities, concentrations etc. and thus might vary widely.

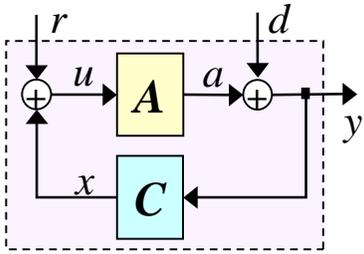
For additional details see
www.cds.caltech.edu/~doyle



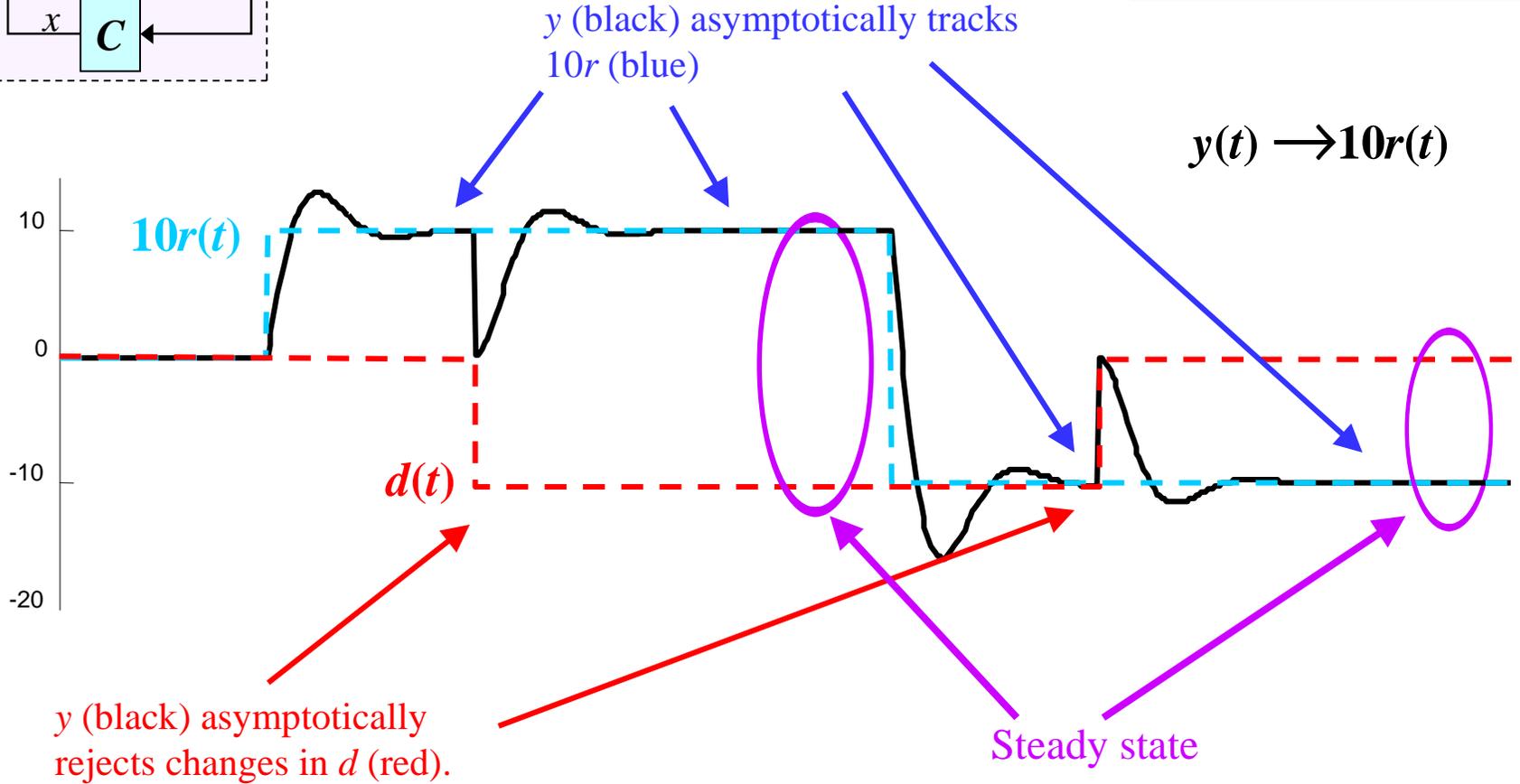
Below is plotted the response of y to some particular inputs r and d with fixed parameters $g=1$, $k_1=0.01$, $k_2=10$, $k_1=0.1$. Note that y (black) asymptotically tracks $10r$ (blue) and rejects changes in d (red). This would be a typical desirable response from a feedback amplifier. We will explore these and other robustness properties of this feedback system.

$$\begin{aligned} \dot{x} &= -k_1 y - k_2 x & y &= d + a \\ \dot{a} &= g u & u &= r + x \end{aligned}$$

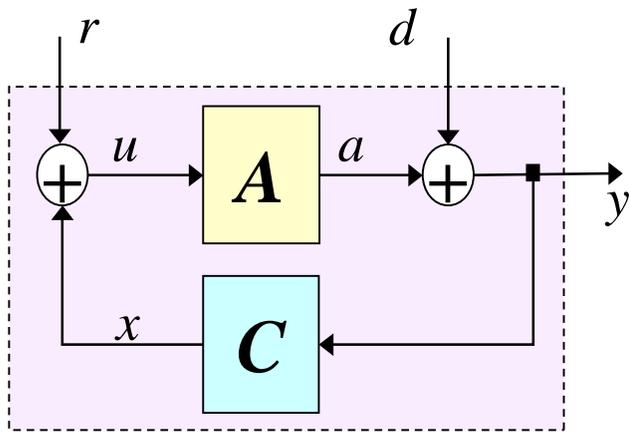




$$\begin{aligned} \dot{x} &= -k_1 y - k_2 x & y &= d + a \\ \dot{a} &= g u & u &= r + x \end{aligned}$$



Initially, we'll focus attention on the steady state (asymptotic) behavior of the system. If the system is stable, then for constant r and d , the other variable will reach a steady state value.



The case of steady state gain here means simply that all variables in Figure 2 (r , d , y , A , C , etc) approach constants, which can be solved for algebraically. That is, after some transient, r and d are held constant, and y too approaches a constant $y=Rr+Sd$. Solving $y=d+ACy+Ar$ gives

$$S \triangleq \frac{1}{1-F} \quad F \triangleq AC$$

$$ACS = \frac{F}{1-F} = \frac{1}{1-F} - 1 = S - 1$$

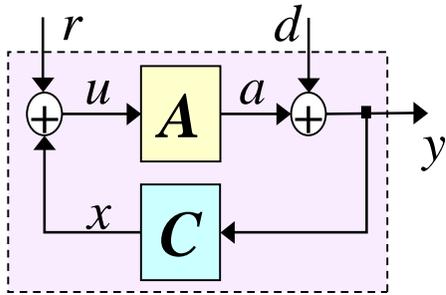
$$y = d + ACy + Ar = d + Fy + Ar$$

$$(1-F)y = Ar + d$$

$$y = ASr + Sd = \frac{1}{C}(S-1)r + Sd$$

$$R \triangleq AS$$

Note: This steady state analysis is at best a “cartoon” of dynamic feedback systems, but helps establish what some of the benefits of feedback are in a simplified setting. It is essential to add dynamics to get a complete picture of feedback control.



$$y = \frac{1}{C}(S-1)r + Sd$$

$$= Ry + Sd$$

$$S \triangleq \frac{1}{1-F} \quad F \triangleq AC$$

$$R \triangleq AS = \frac{1}{C}(S-1)$$

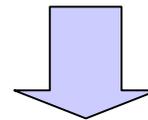
Choosing C small and precise, with A sufficiently large and even sloppy, is one effective, efficient, and robust way to make y a high gain function of r .

Ideally, perfect control would have $|S|=0$, since that gives $y=-r/C$ ($R=-1/C$) completely independent of arbitrary variations in A and d .

If $A \rightarrow \infty$ and $-1/C \gg 1$ then $F \rightarrow -\infty$, $|S| \rightarrow 0$, and $y \rightarrow -r/C$. Then R amplifies r by $-1/C \gg 1$ and is perfectly robust to external disturbance d and to variations in A , provided A is sufficiently large:

$$A \gg -\frac{1}{C} \gg 1 \Rightarrow -F = -AC \gg 1$$

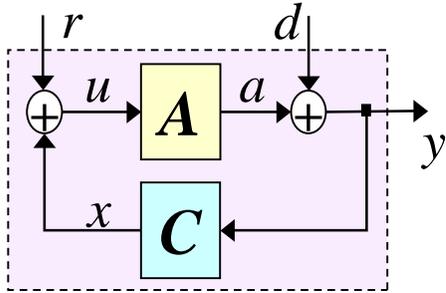
$$\Rightarrow |S| = \left| \frac{1}{1-F} \right| \approx \left| \frac{1}{-F} \right| \ll 1$$



$$y = \frac{1}{C}(S-1)r + Sd = -\frac{1}{C}r + S(d + r/C)$$

$$\approx -\frac{1}{C}r$$

≈ 0



$|S|$ measures the deviation from perfect control, and feedback can attenuate or greatly amplify the effects of uncertainties.

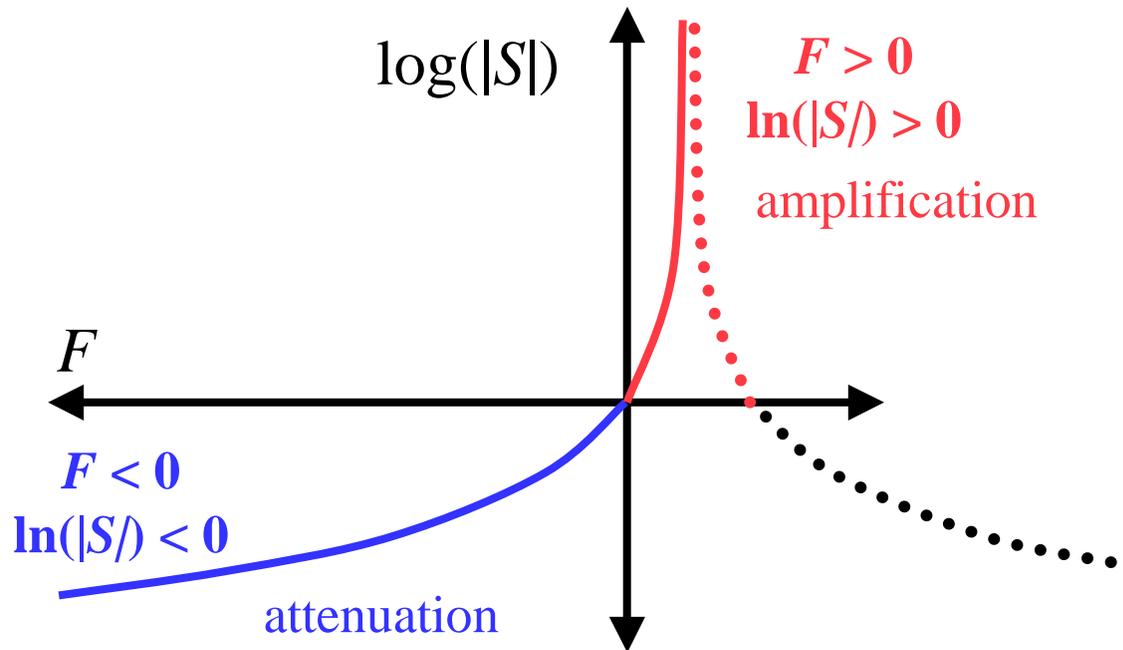
Defining *fragility* as $\log|S|$, note that $F < 0$ iff $|S| < 1$ iff $\log|S| < 0$. $F > 0$ makes $\log|S| > 0$, amplifying d and uncertainty in A , and $F \rightarrow 1$ makes $\log|S| \rightarrow \infty$.

$$y = \frac{1}{C}(S-1)r + Sd$$

$$= Ry + Sd$$

$$S \triangleq \frac{1}{1-F} \quad F \triangleq AC$$

$$R \triangleq AS = \frac{1}{C}(S-1)$$

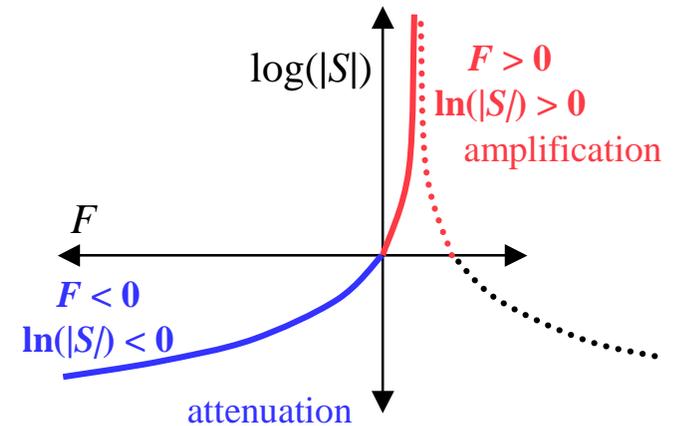


Negative F ($F < 0$) $\Rightarrow \ln(S) < 0 \Rightarrow$ Disturbance attenuated

Positive F ($F > 0$) $\Rightarrow \ln(S) > 0 \Rightarrow$ Disturbance amplified

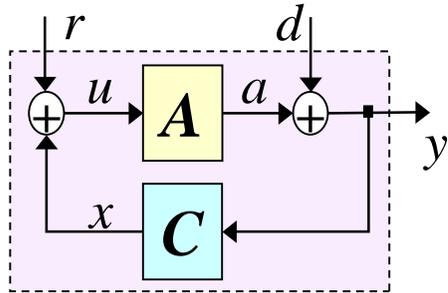
Note: There are many ways in which this steady state analysis can be misleading, and we will explore this next by adding back the simple dynamics, but a few remarks:

- Positive and negative feedback are only well-defined in terms of F for *steady state*, but using $\log(|S|)$ instead will allow generalization to the dynamics case. Thus the widely used terminology of positive and negative feedback is unfortunate, and should probably be discouraged.
- $F > 1$ would typically not be consistent with the existence of a stable steady state, so can be ignored in this part of the story. See note below.



Note: For a system with dynamics which are open loop stable, if $F > 1$ in steady state, then the system dynamics would be unstable in closed loop. This is easily proven using elementary control theory.

Summary so far.



$$y = \frac{1}{C}(S-1)r + Sd$$

$$A \gg -\frac{1}{C} \gg 1 \Rightarrow -F = -AC \gg 1$$

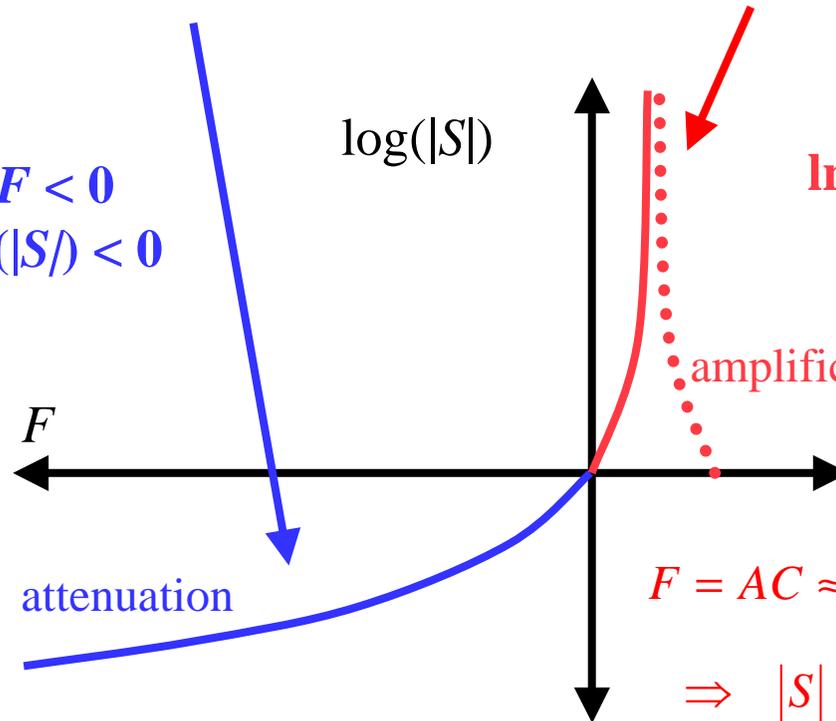
$$\Rightarrow |S| = \left| \frac{1}{1-F} \right| \approx \left| \frac{1}{-F} \right| \ll 1$$

$$\Rightarrow y \approx -\frac{1}{C}r$$

Attenuation of uncertainty in
 A and d .

Feedback can provide extreme
robustness...

$$F < 0 \\ \ln(|S|) < 0$$



...and extreme
fragility.

$$F > 0 \\ \ln(|S|) > 0$$

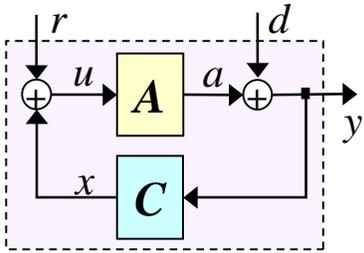
amplification

$$F = AC \approx 1$$

$$\Rightarrow |S| = \left| \frac{1}{1-F} \right| \gg 1$$

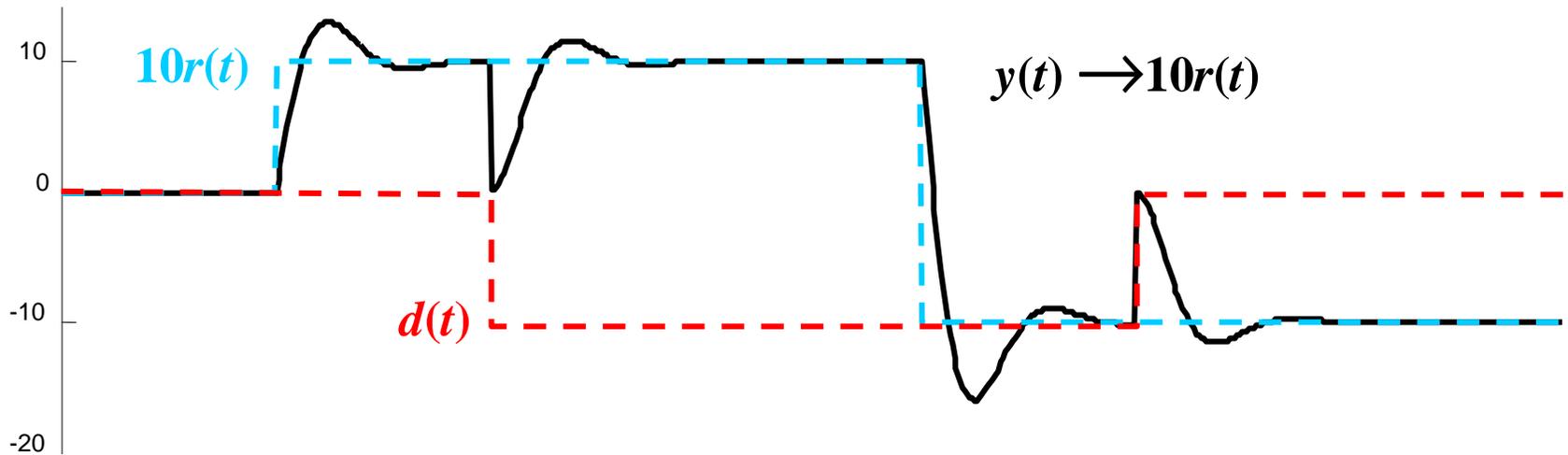
$$\Rightarrow y = S(Ar + d)$$

Amplification of uncertainty
in A and d .



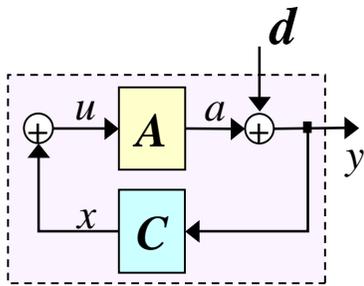
The response $y(t)$ asymptotically tracks $10r(t)$ and rejects changes in $d(t)$. We'll investigate this and other robustness features of this feedback system, but focusing on the dynamics and varying parameters.

$$\begin{aligned} \dot{x} &= -k_1 y - k_2 x & y &= d + a \\ \dot{a} &= g u & u &= r + x \end{aligned}$$



This is a linear system, so the responses to $r + d$ are the sum of independent responses to r and d .

We'll focus on independent responses to unit steps in r and d at $t = 0$ and vary g , k_1 , and k_2 .



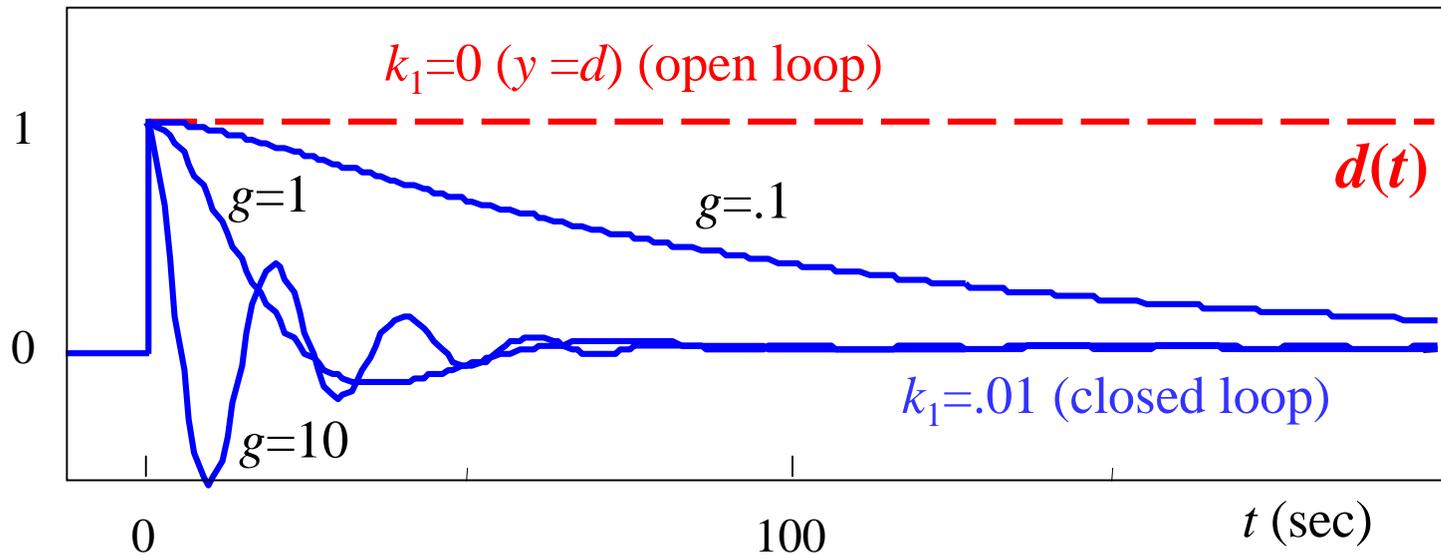
$$\dot{x} = -k_1 y - k_2 x \quad y = d + a$$

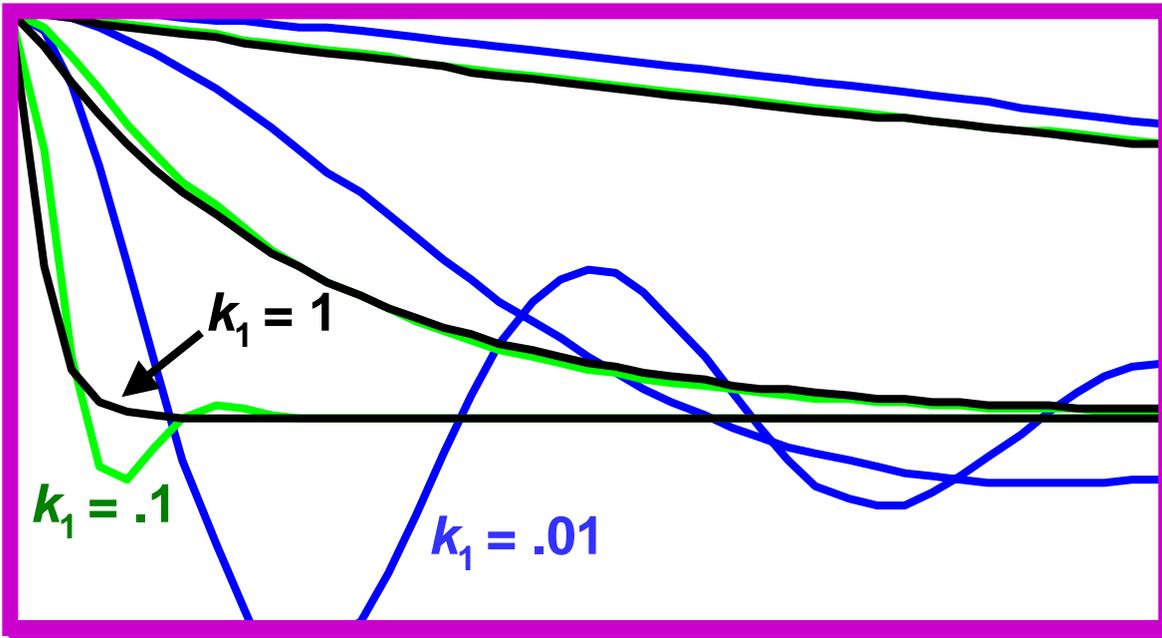
$$\dot{a} = gu \quad u = r + x$$

Response of y to unit step change in d with $r=0$.

Closed ($k_1=.01$, blue) vs. open ($k_1=0$, red) loop response $y(t)$ to unit step change at $t=0$ in $d(t)$ and for

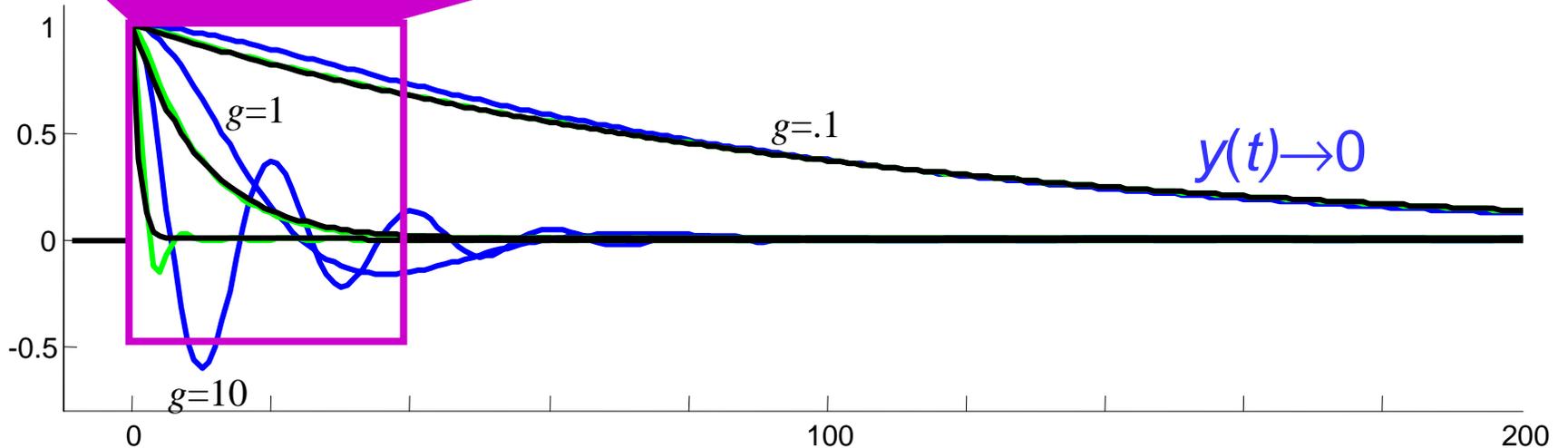
- $g=.1, 1, 10$
- $k_1=.01$
- $k_2=10 k_1$.

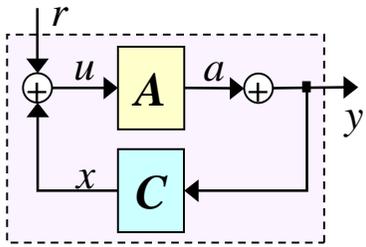




$g = .1, 1, 10,$
 $k_1 = .01, .1, 1$
 $k_2 = 10 k_1.$

$y(t) \rightarrow 0$ for all positive g and k_i . Plot shows simultaneous variation in g and k_i over 2 orders of magnitude. Note that even these large variations preserve stability and steady state tracking.





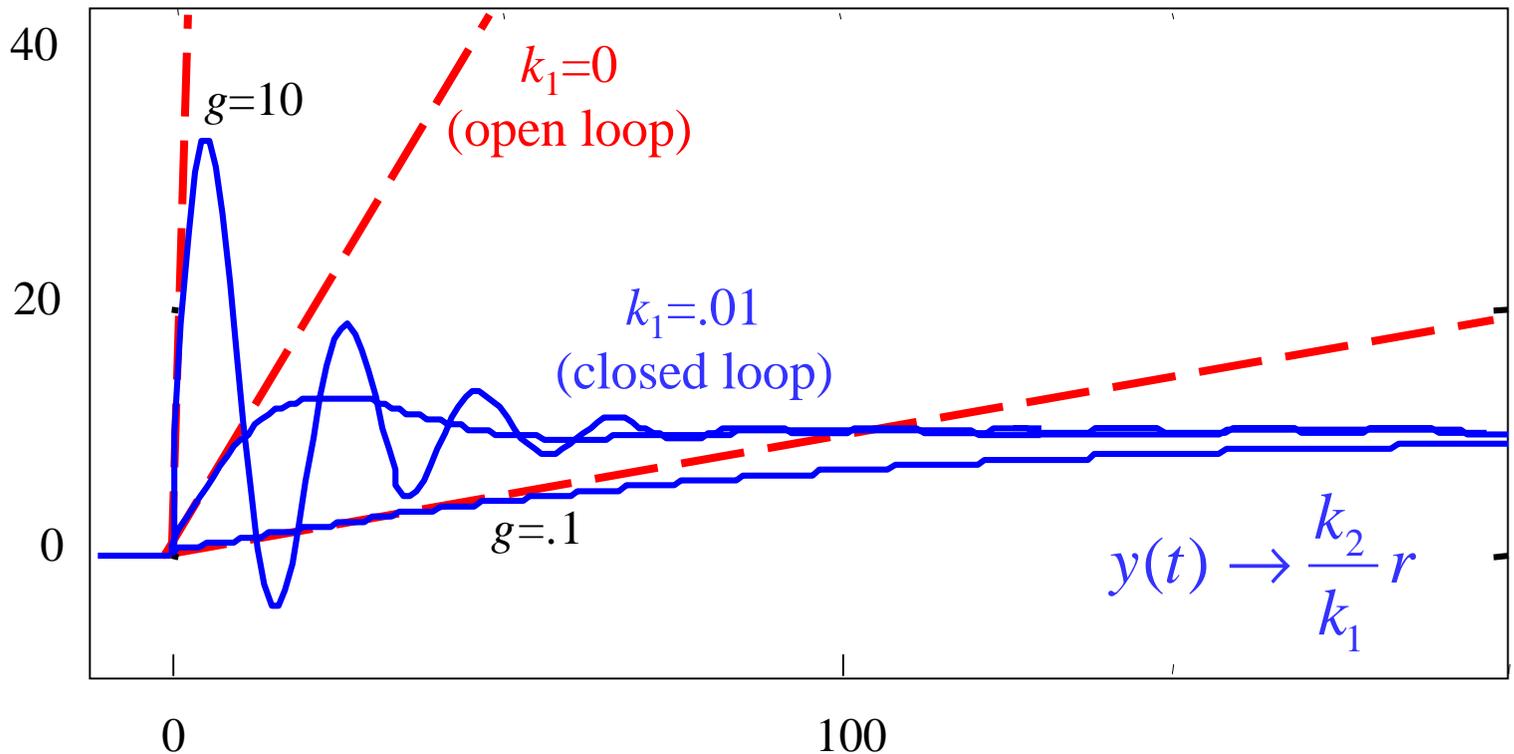
Response of y to unit step change in r with $d=0$.

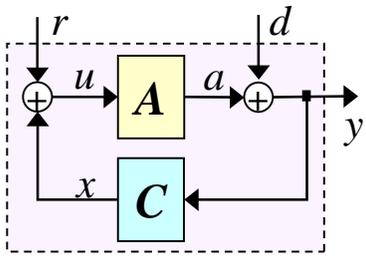
Closed ($k_1=.01$, blue) vs. open ($k_1=0$, red) loop response $y(t)$ to unit step change at $t=0$ in $r(t)$ and for

- $g=.1, 1, 10$
- $k_1=.01$
- $k_2=10 k_1$

$$\dot{x} = -k_1 y - k_2 x \quad y = d + a$$

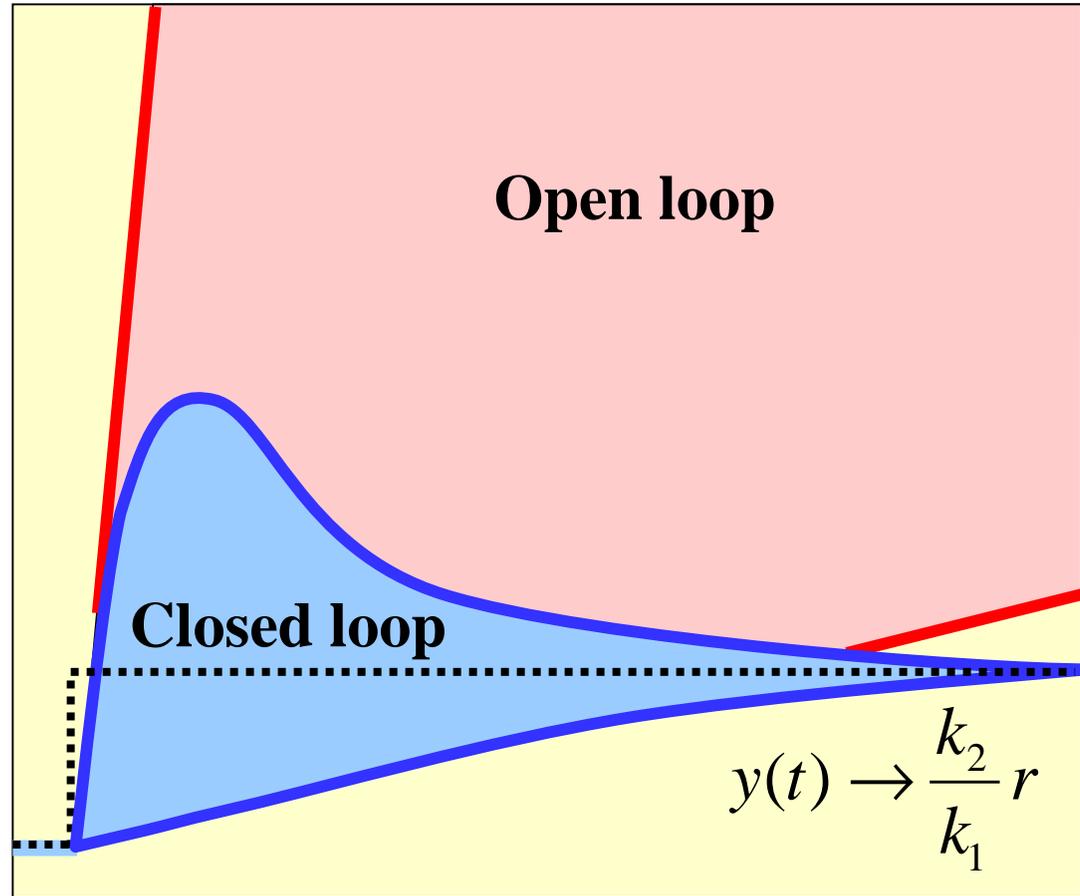
$$\dot{a} = g u \quad u = r + x$$





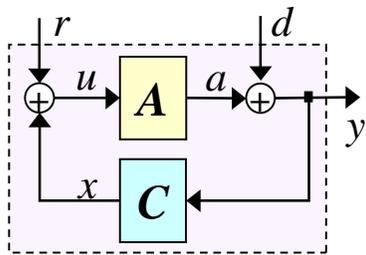
Note that huge variations in **open loop behavior** all lead to the same steady state **closed loop response**.

$$\begin{aligned} \dot{x} &= -k_1 y - k_2 x & y &= d + a \\ \dot{a} &= g u & u &= r + x \end{aligned}$$



$$\frac{k_2}{k_1} r$$

→

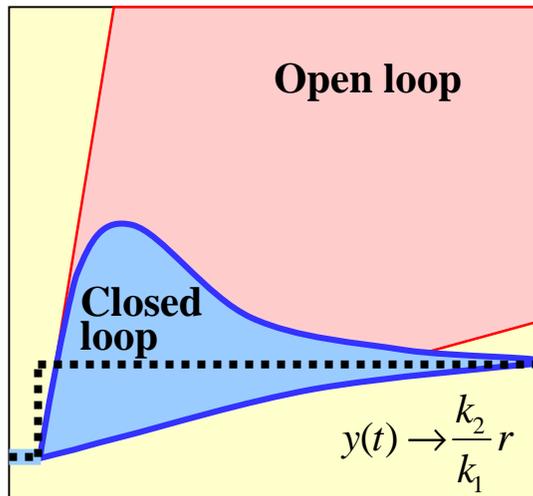


Note that huge variations all lead to the same **steady state closed loop response**.

$$\begin{aligned} \dot{x} &= -k_1 y - k_2 x & y &= d + a \\ \dot{a} &= g u & u &= r + x \end{aligned}$$

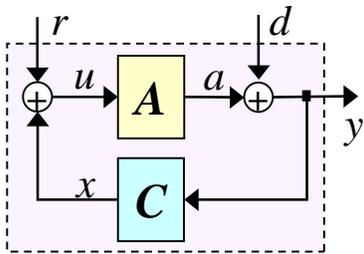
$$\left. \begin{aligned} \text{Steady state} &\Rightarrow \dot{x} = \dot{a} = 0 \\ \text{and } d, r &\text{ constant, so} \\ 0 &= -k_1 y - k_2 x & y &= d + a \\ 0 &= g u & u &= r + x \end{aligned} \right\} \Rightarrow \left\{ \begin{aligned} u &= 0, \\ r &= -x \end{aligned} \right\}$$

$$\Rightarrow y = \frac{k_2}{k_1} r$$



This holds for all values of d and $g > 0$.

Still need to check stability of this equilibrium.



Check stability of this equilibrium:

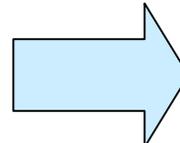
$$y = \frac{k_2}{k_1} r$$

$$\begin{aligned} \dot{x} &= -k_1 y - k_2 x & y &= d + a \\ \dot{a} &= g u & u &= r + x \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} \frac{d}{dt} \\ \end{bmatrix} \begin{bmatrix} x \\ a \\ y \end{bmatrix} = \begin{bmatrix} -k_2 & -k_1 \\ g & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ a \end{bmatrix} + \begin{bmatrix} -k_1 & 0 \\ 0 & g \\ 1 & 0 \end{bmatrix} \begin{bmatrix} d \\ r \end{bmatrix}$$

Stable iff $\text{Re}(\lambda(A)) < 0$ for $A = \begin{bmatrix} -k_2 & -k_1 \\ g & 0 \end{bmatrix}$

$$\det \left(\begin{bmatrix} \lambda + k_2 & k_1 \\ -g & \lambda \end{bmatrix} \right) = \lambda^2 + k_2 \lambda + g k_1$$

$$\Rightarrow \lambda = \frac{-k_2 \pm \sqrt{k_2^2 - 4gk_1}}{2}$$



Stable
iff
 $k_2 > 0$ and $gk_1 > 0$

Summary:

$$y = \frac{k_2}{k_1} r \text{ stable equilibrium}$$

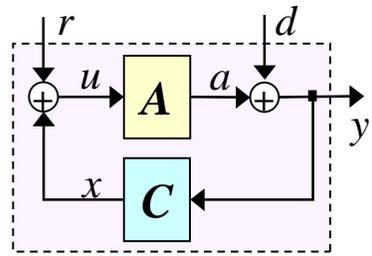
iff

$$k_2 > 0 \text{ and } gk_1 > 0$$

Steady state:

$$x = -r \quad y = \frac{k_2}{k_1} r$$

$$a = \frac{k_2}{k_1} r - d \quad u = 0$$



Important idea:

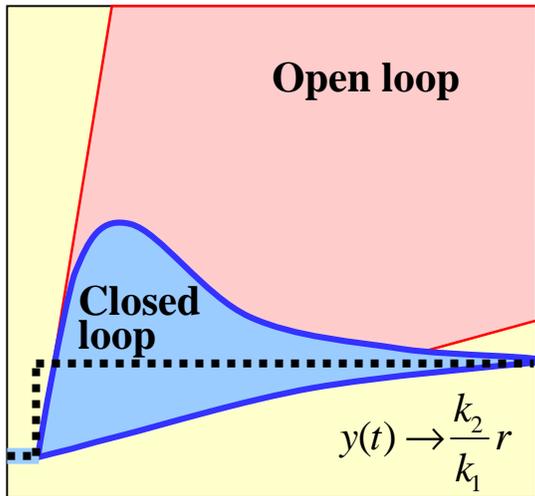
$$\dot{a} = gu \Rightarrow a(t) = a(0) + g \int_0^t u(\tau) d\tau \quad \text{"integral feedback"}$$

$$\begin{aligned} \dot{x} &= -k_1 y - k_2 x & y &= d + a \\ \dot{a} &= gu & u &= r + x \end{aligned}$$

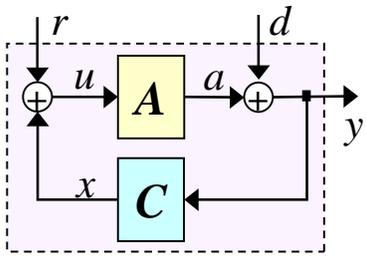
Stable iff
 $k_2 > 0$ and $gk_1 > 0$

Steady state $\frac{da}{dt} = 0 \Rightarrow \{u = 0\}$
 $\Rightarrow \{x = -r\} \Rightarrow \left\{ y = \frac{k_2}{k_1} r \right\}$

Independent of constant d or $g > 0$.



Integral feedback is **necessary** and sufficient for perfect steady state tracking of reference r . (Necessity requires more complex proof.)



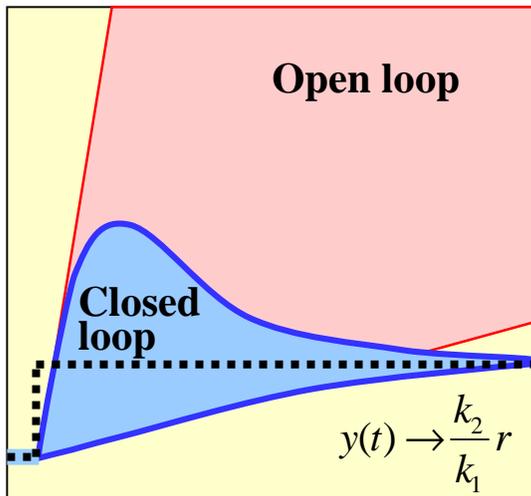
$$y \rightarrow \frac{k_2}{k_1} r$$

$$\begin{aligned} \dot{x} &= -k_1 y - k_2 x & y &= d + a \\ \dot{a} &= g u & u &= r + x \end{aligned}$$

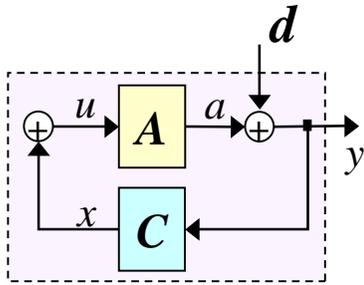
- This holds for all constant values of d and $g > 0$.
- High closed loop gain depends only on the ratio k_2/k_1 but does not otherwise depend on any of the individual parameter values.
- In both engineering and biochemical systems it is possible to make ratios such as k_2/k_1 much less uncertain than individual parameters k_1 and k_2

For constant d and r , 3 things can happen:

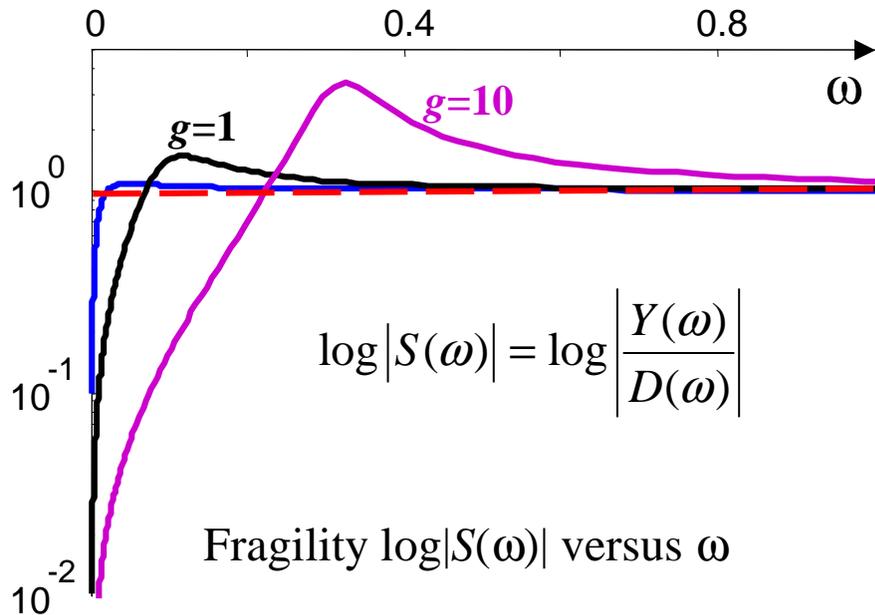
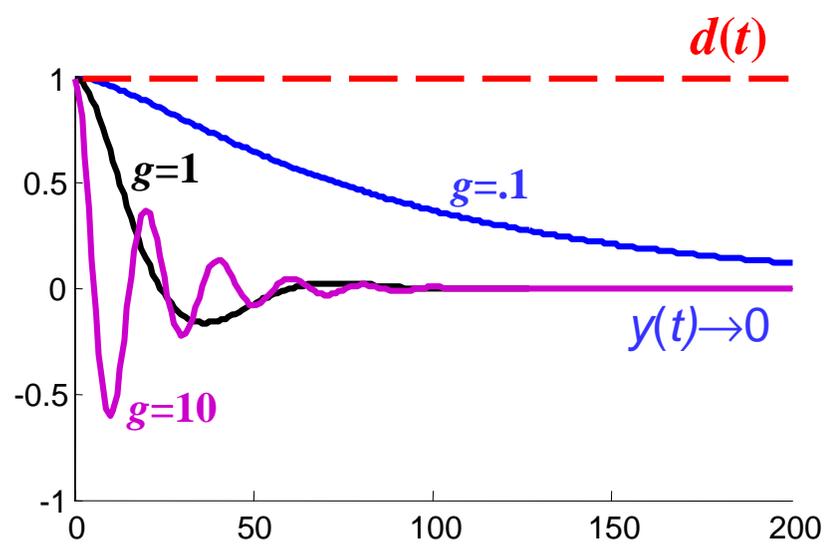
1. No steady state is reached (unstable).
2. Steady state (stable).
3. Zero steady state error.



For zero steady state error in closed loop, it is necessary and sufficient that in open loop, the system have a certain kind of instability, i.e. integral feedback.



Fragility enters in the transient response. When g is increased, the response is faster but oscillatory .

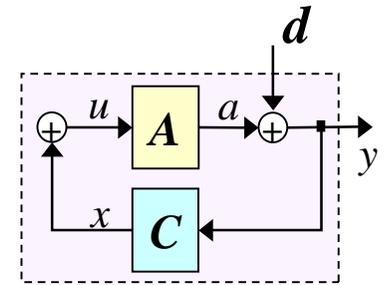
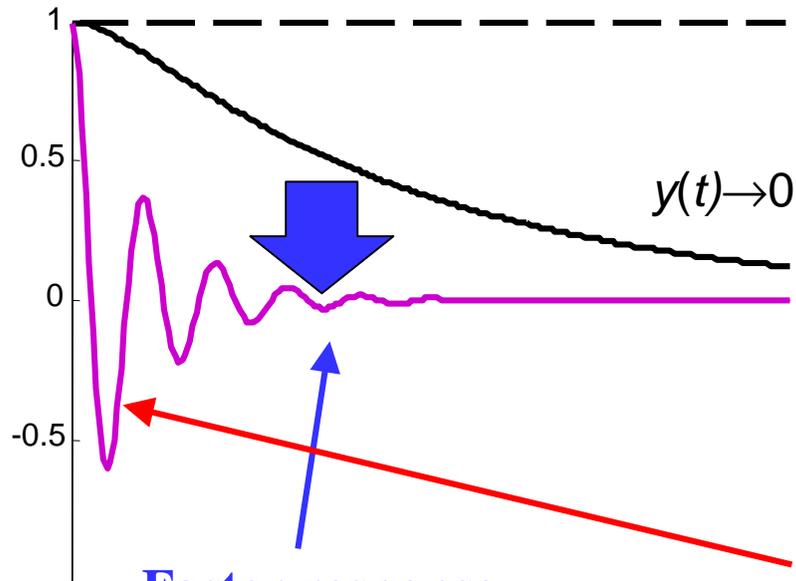


$$\log |S(\omega)| = \log \left| \frac{Y(\omega)}{D(\omega)} \right|$$

Fragility $\log|S(\omega)|$ versus ω

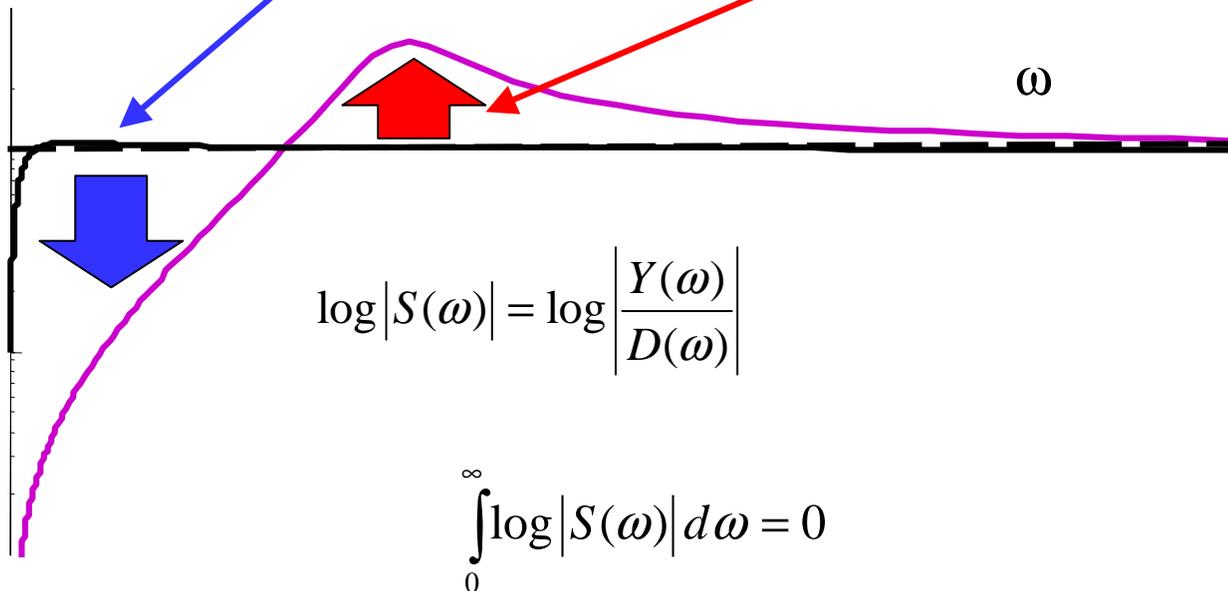
For increasing g , low frequency robustness ($\log|S(\omega)| < 0$) is improved but at the expense of increased fragility ($\log|S(\omega)| > 0$) at higher frequencies. In fact, it can be proven that for all g

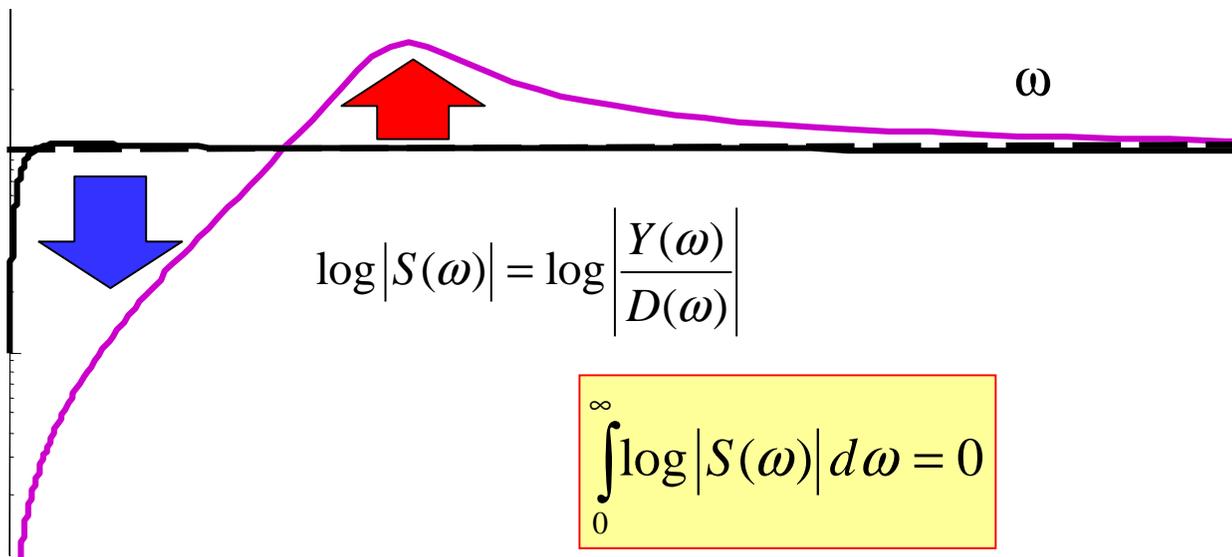
$$\int_0^{\infty} \log |S(\omega)| d\omega = 0$$



Faster response

Transient oscillations





- Net fragility is, in this sense, a conserved quantity. Robustness ($\log|S(\omega)| < 0$) is paid for by an equal fragility ($\log|S(\omega)| > 0$) which amplifies d and uncertainty in A . This quite general result also holds for arbitrary parameters, control systems, and disturbances. Thus there are always nonconstant (e.g. sinusoidal) $d(t)$ that would be *amplified* in $y(t)$.
- For sufficiently large g the frequency domain peak and time domain transients become unacceptably large, though still stable.
- One interpretation is that negative feedback is always balanced by an equal and opposite positive feedback. Strictly speaking, with dynamics this is not well defined, and $\log|S(\omega)|$ gives the correct generalization.
- Relatively rare circumstances can involve an inequality (\geq). This is worse, but means that this is an inequality constraint rather than a pure “conservation” law.
- This is a standard result in control theory, and the proof needs only advanced undergraduate complex variables theory, involving a contour integral of $\log(S(\omega))$.
- More complex controllers provide more subtle manipulation, but do not avoid, this tradeoff.
- The spiraling complexity in advanced biological organisms is largely due to greater sophistication in managing this tradeoff.