Nonlinear Systems

The behavior of linear systems is fairly well understood.

For treatment of linear systems, it helps that we know how to solve ordinary linear differential equations analytically.

More importantly, analytical tools are well developed: convolution, Laplace Transform (Fourier Transform), Linear Algebra, Linear Operator Theory, etc.

Nonlinear systems are not as well understood.

And the fact that we cannot solve nonlinear differential equations analytically does not help.

Unfortunate fact #1: nonlinear systems are by far more interesting than linear system.

Unfortunate fact #2: virtually all systems in nature are nonlinear.

Linear systems: $$y(t) = L\left[t; u_{[-\infty,t]}\right] = \underbrace{\int_{-\infty}^{t} g(t-\tau)u(\tau)d\tau}_{\text{convolution}}$$

Nonlinear systems: $$y(t) = N\left[t; u_{[-\infty,t]}\right] = \underbrace{\sum_{n=1}^{\infty} F_n[u]}_{\text{Volterra series}}$$

where:

$$F_1[u] = \int_{-\infty}^{t} h_1(t-\tau_1)u(\tau_1)d\tau_1$$
$$F_2[u] = \iint_{-\infty}^{t} h_2(t-\tau_1, t-\tau_2)u(\tau_1)u(\tau_2)d\tau_1 d\tau_2$$
$$F_3[u] = \iiint_{-\infty}^{t} h_3(t-\tau_1, t-\tau_2, t-\tau_3)u(\tau_1)u(\tau_2)u(\tau_3)d\tau_1 d\tau_2 d\tau_3$$
$$\vdots$$

h_n – impulse response of order n
$h_1 = g$ (impulse response)

Some Distinct Properties of Nonlinear Systems

1) Frequency Mixing

$$\sin(\omega t) \longrightarrow \boxed{\text{LTI system}} \longrightarrow \underbrace{A(\omega)\sin(\omega t + \phi(\omega))}_{\textbf{steady state}}$$

$$\sin(\omega t) \longrightarrow \boxed{\text{Nonlinear system}} \longrightarrow$$

ω

$2\omega, 3\omega, 4\omega, \cdots$ - **(super)harmonics**

$\frac{\omega}{2}, \frac{\omega}{3}, \frac{\omega}{4}, \cdots$ - **subharmonics**

"I want to contrast what we are doing with what is done in linear networks. In linear networks, the standard input has been a trigonometric input. That comes from the fact that if we put two trigonometric inputs into a linear network, the outputs add. We can study each separately, and they do not mix. In a nonlinear network, they do mix, and we get no great advantage by bringing in trigonometric functions."

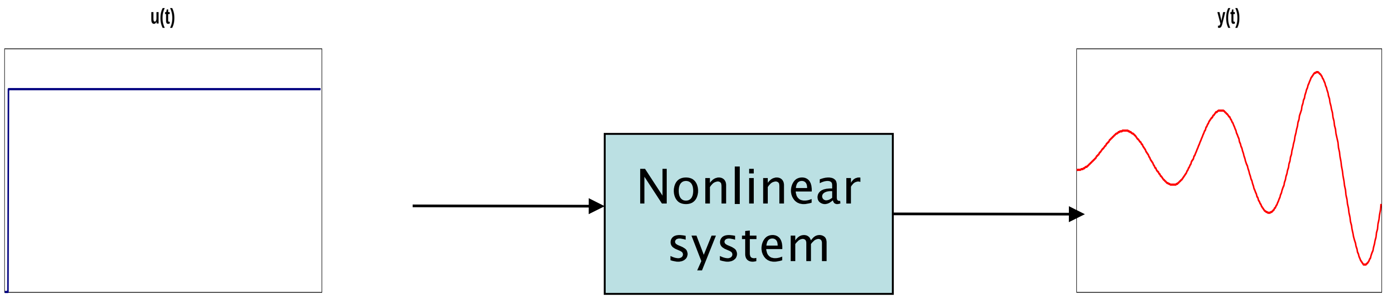
N. Wiener, Nonlinear Problems in Random Theory, 1949.

2) Intermodulation Distortion

$$\sin(\omega_1 t) + \sin(\omega_2 t) \longrightarrow \boxed{\text{Nonlinear system}} \longrightarrow \omega_1, \omega_2, 2\omega_1, 2\omega_2, \omega_1 - \omega_2, \omega_1 + \omega_2$$

E.g. $y(t) = a\,u(t) + b\,u^2(t)$ – static nonlinearity

3) Resonant Jump Phenomenon



Canonical 2nd order system: $\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2 y(t) = k\omega_n^2 u(t)$

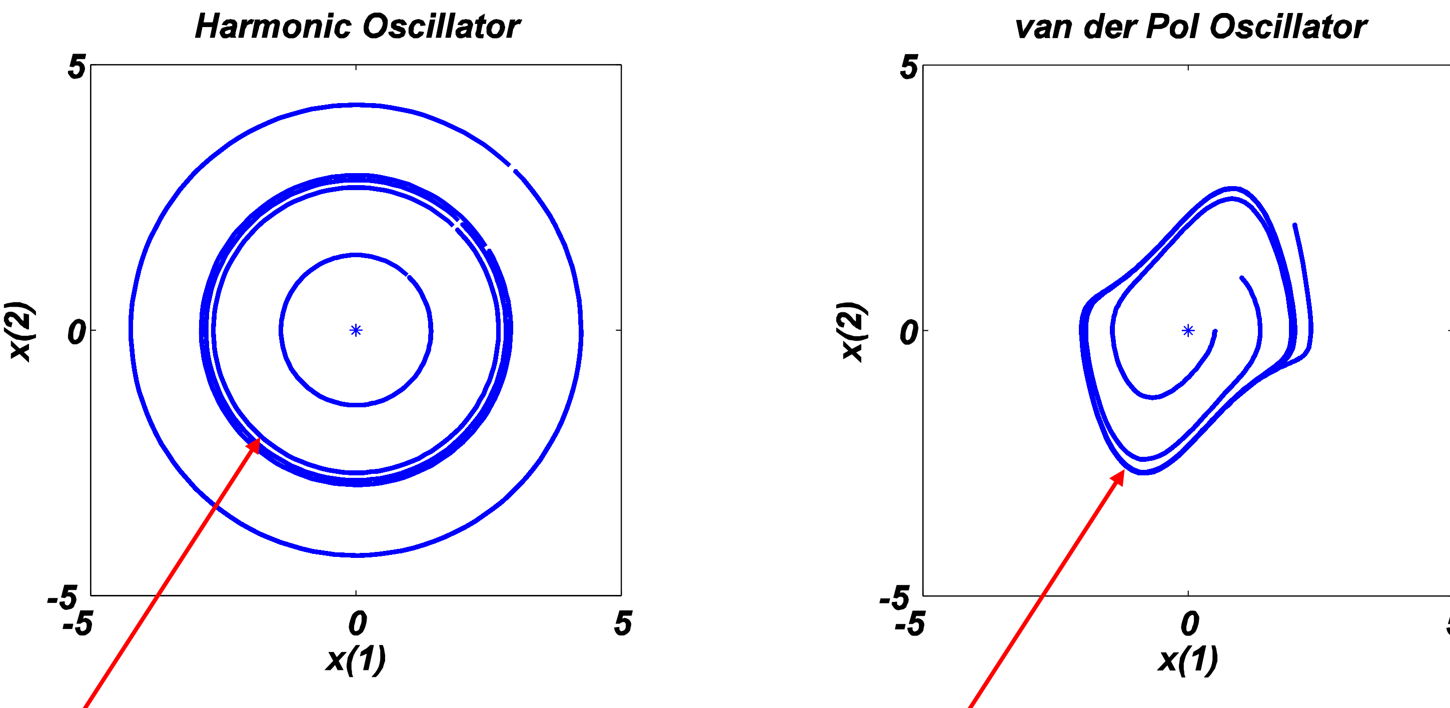
Note: ζ property of the system.

Nonlinear system ($\omega_n = 1, k = 1$): $\ddot{y}(t) + 2\underbrace{[1 - u(t)]}_{\zeta}\dot{y}(t) + y(t) = u(t)$

Note: ζ depends on $u(t)$.

Play with `resonant_jump.m`

4) Limit Cycle Phenomenon: The existence of an isolated closed trajectory in the phase space.



closed but not isolated

closed and isolated

$$\ddot{y}(t) + \mu(y^2(t) - 1)\dot{y}(t) + y(t) = 0$$

note: μ is a parameter

Harmonic oscillator (LTI system) can be recovered from the van der Pol oscillator ($\mu = 0$)*. Play with `vanderpol_oscillator.m`

5) Finite Escape Time: For linear systems, if a response is going to blow-up, it does so in an asymptotic fashion (as t goes to infinity).

For unstable systems (at least one pole with positive real part) we have

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} e^{\overbrace{\lambda}^{>0} t}(\cdots) \to \infty$$

However, for any finite t, the components of $x(t)$ are still finite. For nonlinear systems, this may not be true.

$$\dot{y}(t) = y^2(t)$$

Solution*:

$$y(t) = \frac{y(t_0)}{1 - (t - t_0)y(t_0)}$$

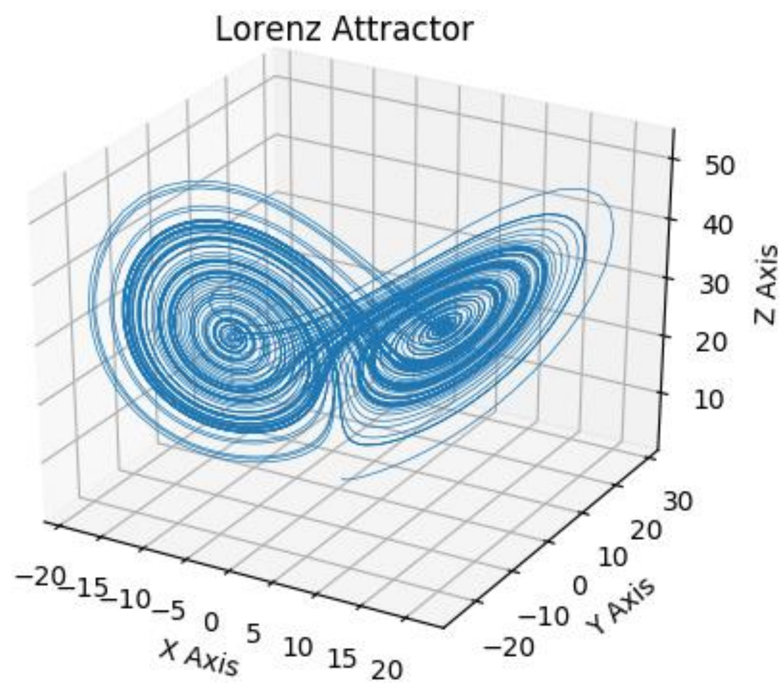
6) Strange Attractor/Chaos: In linear systems, attractors are (equilibrium) points. In nonlinear systems, in addition to equilibrium points and limit cycles, other types of attractors may arise such as strange attractors.

Example: [Lorenz system—chaos]

$$\dot{x}_1(t) = \sigma(x_2(t) - x_1(t))$$
$$\dot{x}_2(t) = x_1(t)\big(\rho - x_3(t)\big) - x_2(t)$$
$$\dot{x}_3(t) = x_1(t)x_2(t) - \beta x_3(t)$$

$\sigma > 0$ – Prandtl number
$\rho > 0$ – Rayleigh number
$\beta > 0$ – nameless parameter

If $\sigma = 10, \rho = 28, \beta = 8/3$, the system exhibits chaotic behavior!



Example: [Barnsley's fern—randomly generated fractal]

$$
\begin{aligned}
i &= 1,2,3,4 \\
P(1) &= 0.01, P(2) = 0.85, P(3) = 0.07, P(4) = 0.07 \\
A_1 &= \begin{bmatrix} 0.00 & 0.00 \\ 0.00 & 0.16 \end{bmatrix}, A_2 = \begin{bmatrix} 0.85 & 0.04 \\ -0.04 & 0.85 \end{bmatrix}, A_3 = \begin{bmatrix} 0.20 & -0.26 \\ 0.23 & 0.22 \end{bmatrix}, A_4 = \begin{bmatrix} -0.15 & 0.28 \\ 0.26 & 0.24 \end{bmatrix} \\
u_1 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 0.00 \\ 1.60 \end{bmatrix}, u_3 = \begin{bmatrix} 0.00 \\ 1.60 \end{bmatrix}, u_4 = \begin{bmatrix} 0.00 \\ 0.44 \end{bmatrix} \\
x_{n+1} &= A_i x_n + u_i
\end{aligned}
$$