Phase Portraits of 2-D LTI Systems

$$\dot{x}(t) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} x(t) \qquad x(0) = x_0$$

Solution: $\boxed{x(t) = e^{At}x_0}$ $\quad e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$

$x(t)$ – depends on the eigenvalues of A!

$$\det(sI - A) = 0$$

$$s_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$$

where $\tau = a + d = \mathrm{trace}(A)$ and $\Delta = ad - bc = \det(A)$

Case I Complex-Conjugate Poles ($s_{1,2} \in \mathbb{C}^{1\times1}$)

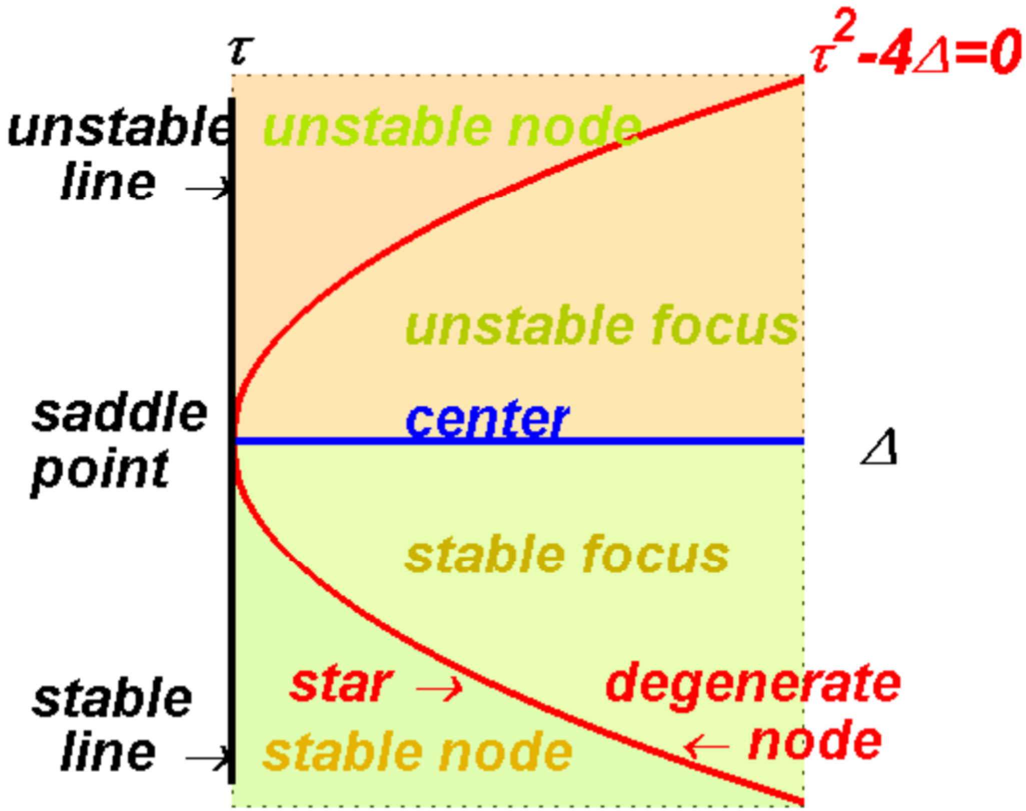
- **stable focus (spiral)**
- **unstable focus (spiral)**
- center (marginally stable)

Case II Real Poles ($s_{1,2} \in \mathbb{R}^{1\times1}$)

case i	$0 > s_1 > s_2$	(**stable node**)
case ii	$s_1 > s_2 > 0$	(**unstable node**)
case iii	$s_1 > 0 > s_2$	(**saddle point**)
case iv	$s_1 > s_2 = 0$	(unstable line)
case v	$0 = s_1 > s_2$	(stable line)
case vi	$s_1 = s_2 > 0$ (2 lin. ind. eigenvec.)	(**unstable star**)
	$s_1 = s_2 < 0$	(**stable star**)
case vii	$s_1 = s_2 > 0$ (1 lin. ind. eigenvec.)	(**unstable degenerate node**)
	$s_1 = s_2 < 0$	(**stable degenerate node**)
case viii	$s_1 = s_2 = 0$ (outrageously trivial)	

Play with `equilibrium_points.m`

Cases shown in **blue** are called ***hyperbolic equilibria***.



So why are we studying phase portraits of 2nd order LTI systems?

It turns out that under certain conditions, one can study the phase portrait of a nonlinear system, by examining the phase portrait of a linear system.

This leads us to the concept of linearization.

Linearization

Assume a nonlinear time-invariant (NLTI) system: $\dot{x}(t) = f(x(t))$

with an equilibrium point: $x^\star \ (f(x^\star) = 0)$

Let us linearize the vector field $f(x)$ around $x^\star$:

Note: $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$ → state of n-dimensional dynamic system

$$f\big(x(t)\big) = \begin{bmatrix} f_1(x_1(t), x_2(t), \cdots, x_n(t)) \\ f_2(x_1(t), x_2(t), \cdots, x_n(t)) \\ \vdots \\ f_n(x_1(t), x_2(t), \cdots, x_n(t)) \end{bmatrix}$$ → n nonlinear functions

Short notation: $f\big(x(t)\big) = \begin{bmatrix} f_1(x(t)) \\ f_2(x(t)) \\ \vdots \\ f_n(x(t)) \end{bmatrix}$

$$\underbrace{\frac{dx(t)}{dt} = f\big(x(t)\big)}_{\text{NLTI system}} \xrightarrow{\text{linearization}} \underbrace{\frac{dz(t)}{dt} = Az(t)}_{\text{LTI system}}$$

Major questions:

Q1: By analyzing the stability of a linearized model (LTI system), what can be said about the stability of the original (NLTI) system? E.g. if the linearized model is stable, is the nonlinear model stable too?

Q2: By analyzing the type of equilibria of linearized model, what can be said about the type of equilibria of the original (NLTI) system? E.g. if the linearized equilibrium is of a certain type, say a node, is the equilibrium of the original system is also a node.

$dx(t)/dt = f(x(t))$ find equilibrium $x^\star$ such that $f(x^\star) = 0$.

Linearize $f(x)$ around $x^\star$ using Taylor series expansion.

This can be done for any number of dimensions n.

$$dx(t)/dt = Df(x^\star)\,[x(t) - x^\star] + H.O.T.$$

$$z(t) := x(t) - x^\star$$

$$dz(t)/dt = Df(x^\star)\,z(t) = A\,z(t) \quad (H.O.T.\,neglected)$$

$Df(x^\star)$ – Jacobian matrix ($n\ x\ n$)

Q1: By analyzing $dz(t)/dt = A\,z(t)$, what can be said about the stability of the system $dx(t)/dt = f(x(t))$?

If $z^\star$ is a hyperbolic equilibrium (no eigenvalue of *A* has zero real part), we say that the stability of $z^\star$ determines the stability of $x^\star$, and in turn, the stability of the nonlinear system $dx(t)/dt = f(x(t))$.
If $z^\star$ is a non-hyperbolic equilibrium, then nothing can be said about stability (H.O.T. are important).

If we restrict ourselves to $n = 2$ (2-D or 2nd order systems) even stronger results are possible but conditions are more restrictive.

Q2: By analyzing the phase portrait of $dz(t)/dt = A\,z(t)$, what can be said about the phase portrait of the system 2-D nonlinear system:
$dx(t)/dt = f(x(t))$ (at least locally around $x^\star$)?

As long as $z^\star$ is not one of the borderline cases (line, center, star, degenerate node), i.e. $z^\star$ is a saddle point, node or focus, $x^\star$ is also a saddle point, node or focus. In other words the phase portraits of $dz(t)/dt = A\,z(t)$ and $dx(t)/dt = f(x(t))$ are qualitatively similar (at least locally around $x^\star$).

If $z^\star$ is one of the borderline cases, H.O.T. are important and cannot be neglected.

Example: van der Pol Oscillator (we will demonstrate both of these points).

$$\ddot{y}(t) + \mu(y^2(t) - 1)\dot{y}(t) + y(t) = 0$$

Play with `linearized_vanderpol.m`