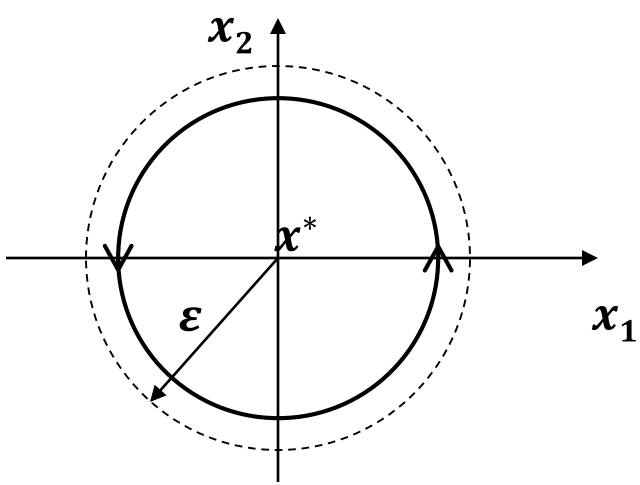
From last time:

If an equilibrium x^* is Lyapunov stable it need not be attractive. An equilibrium that is Lyapunov stable, but not attractive is called marginally (neutrally) stable. Examples are numerous. Any system that has center as its equilibrium.

Example: center.



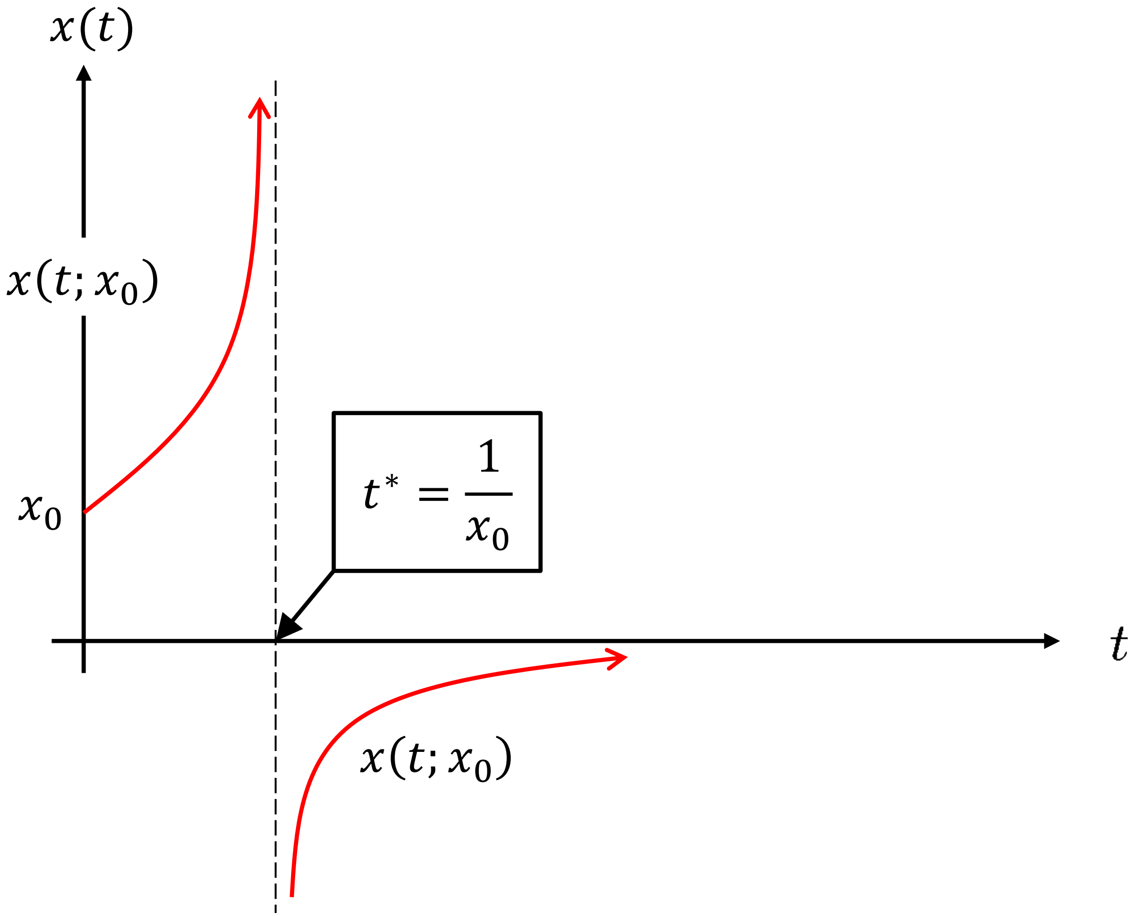
$$\delta = \varepsilon$$

$$\left\| x(t; x_0) - \underbrace{x^*}_{0} \right\| < \varepsilon$$

$$\forall x_0 \in \mathbb{R}^n \colon \|x_0\| < \delta$$

If an equilibrium x^* is attractive, it need not be Lyapunov stable. Examples are not so numerous, but here is one:

$$\dot{x}(t) = x^2(t) \qquad\qquad x(t;x_0) = \frac{x_0}{1 - tx_0}$$



Isoclines

Recall: $\underbrace{dx(t)/dt}_{\text{velocity}} = f(x(t))$

Also note that $f\big(x(t)\big) \in \mathbb{R}^n$, therefore $f(x(t))$ is a vector field.

Since $f(x(t))$ is parameterized by t, it is often called the flow of the vector field.

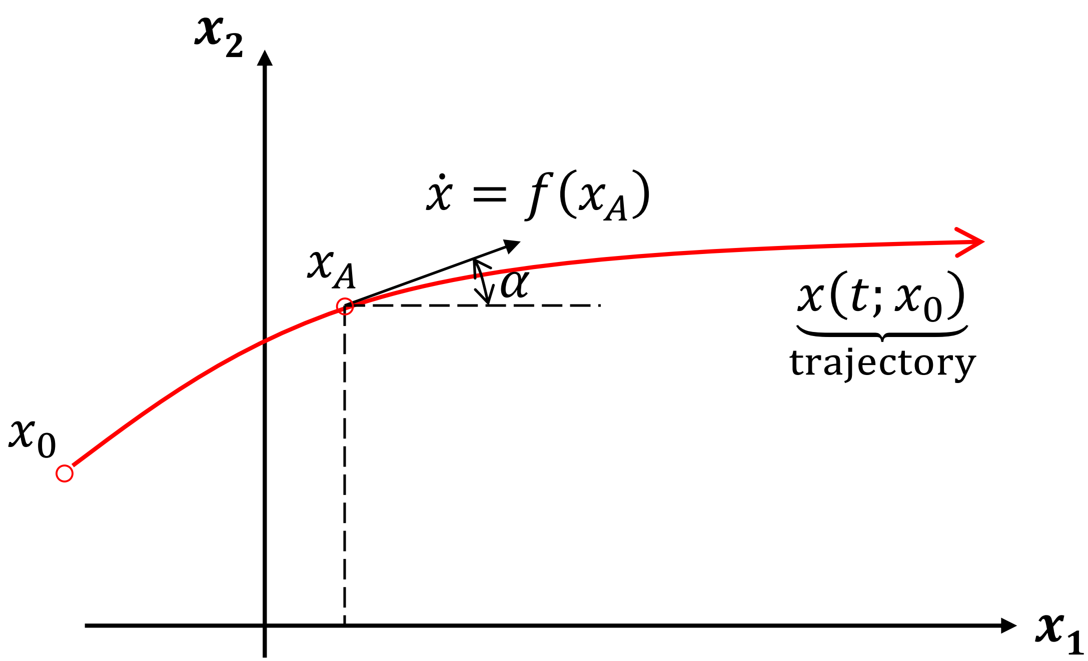
Special case: $n = 2$

$$\frac{dx(t)}{dt} = f\big(x(t)\big)$$

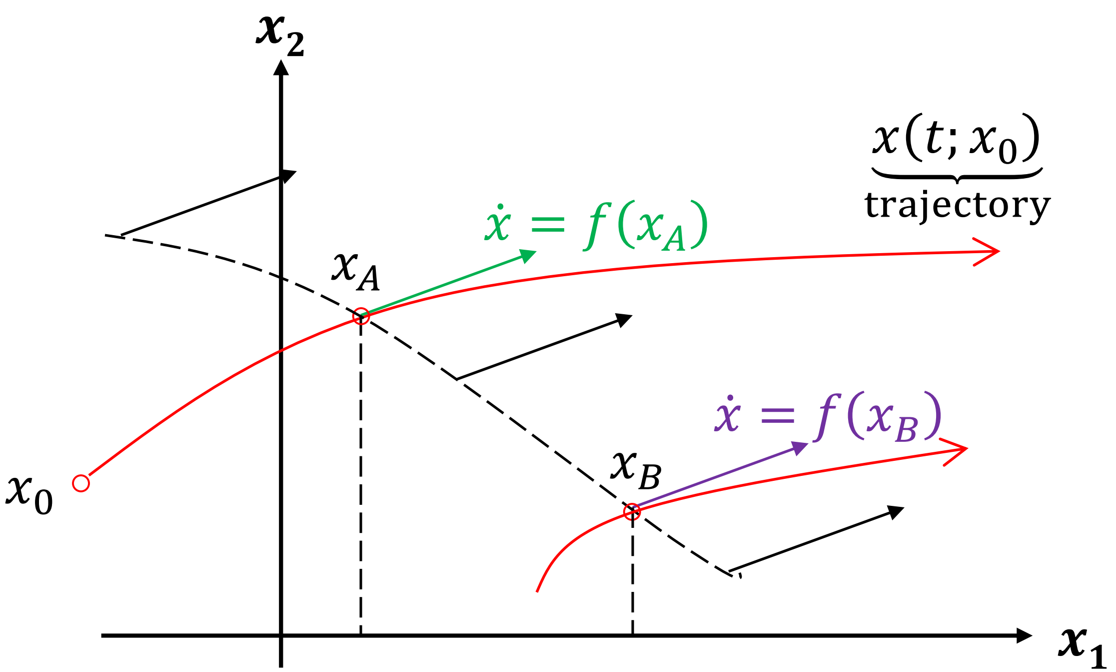
$$x(t) \in \mathbb{R}^2$$

$$x(0) = x_0$$

$\tan\alpha =$ slope of the vector field at $x = x_A$



What if I find another point x_B such that: $slope\ @\ x_B\ =\ slope\ @\ x_A$?



Then x_B and x_A belong to a curve called <u>isocline</u>.

A collection of states x such that $Slope\big(f(x)\big) = Const. = C$ is called an isocline.

Note: isocline is not necessarily a trajectory.

What are they useful for?

Isoclines are useful for sketching phase portraits of 2-D systems.

They are also easy to calculate.

Slope of $f(x)$ is determined by:

$$\text{Slope} = \text{Const.} = C$$

$$\text{Slope} = \frac{f_2(x)}{f_1(x)} = \frac{\dot{x}_2}{\dot{x}_1} = \frac{\frac{dx_2}{dt}}{\frac{dx_1}{dt}} = \frac{dx_2}{dx_1}$$

$$\frac{dx_2}{dx_1} = \underbrace{C = \frac{f_2(x)}{f_1(x)}}_{\text{eq. of isocline}}$$

Example: $$\frac{d^2y(t)}{dt^2} - 0.5\, y(t)\frac{dy(t)}{dt} + y(t) = 0$$

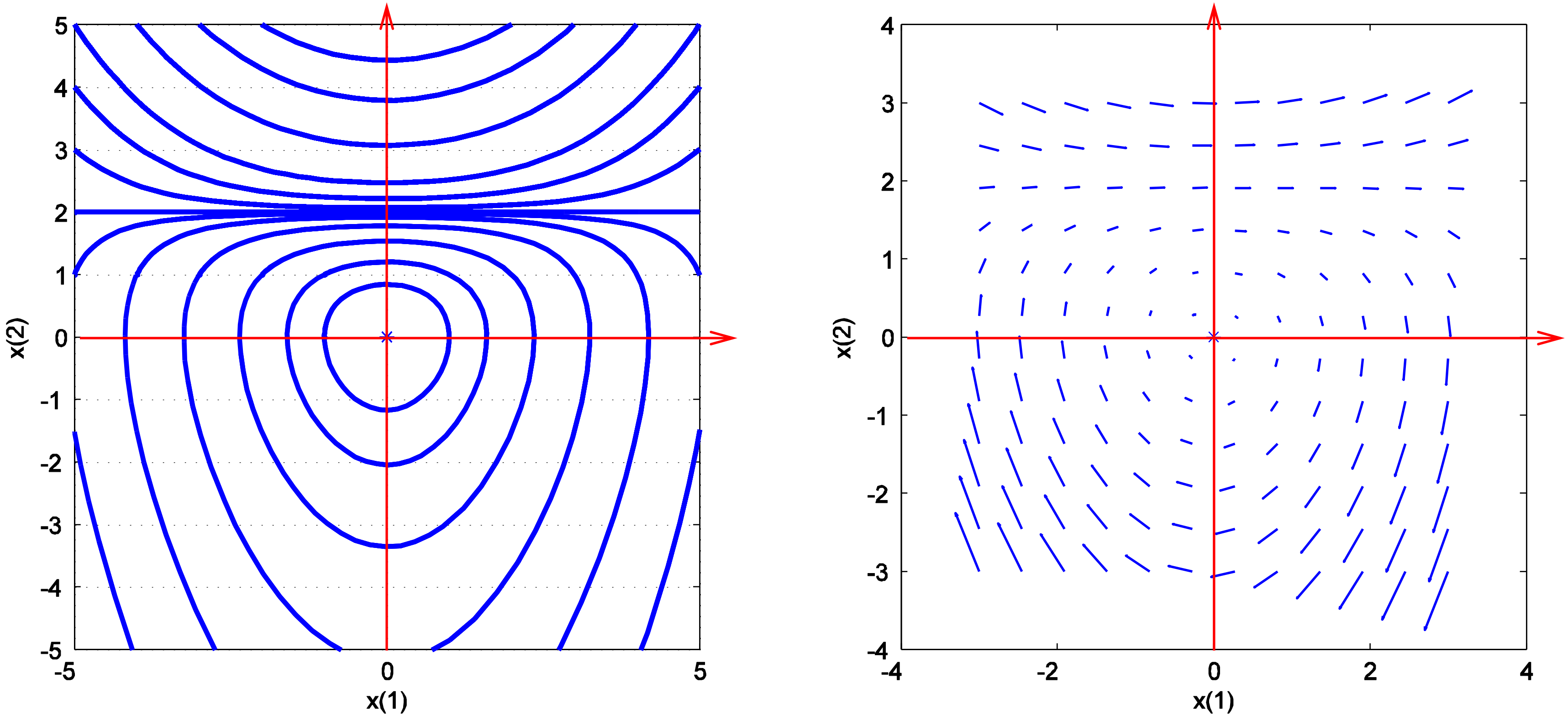
$$x_1(t) \coloneqq y(t)$$
$$x_2(t) \coloneqq \frac{dy(t)}{dt}$$

$$\frac{dx_1(t)}{dt} = x_2(t)$$
$$\frac{dx_2(t)}{dt} = 0.5x_1(t)x_2(t) - x_1(t)$$

Isocline: $\frac{\dot{x}_2}{\dot{x}_1} = \frac{dx_2}{dx_1} = \boxed{\frac{0.5x_1x_2 - x_1}{x_2} = C}$ implicit equation

Similarly: $x_2(C - 0.5x_1) = -x_1 \Rightarrow \boxed{x_2} = -\frac{x_1}{C-0.5x_1} = \frac{x_1}{0.5x_1 - C} = \boxed{\frac{2x_1}{x_1 - 2C}}$

explicit equation



x(1)
x(2)

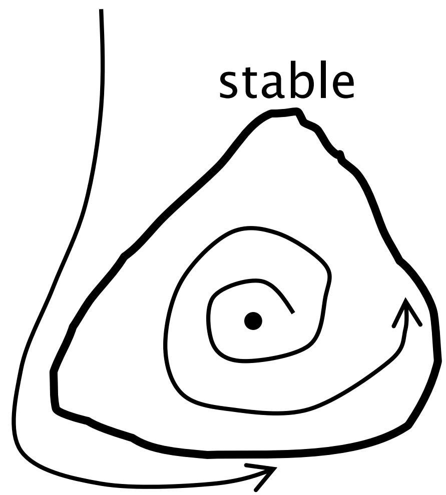
Limit Cycle

Limit cycle (periodic orbit) is an isolated closed trajectory in the phase space.

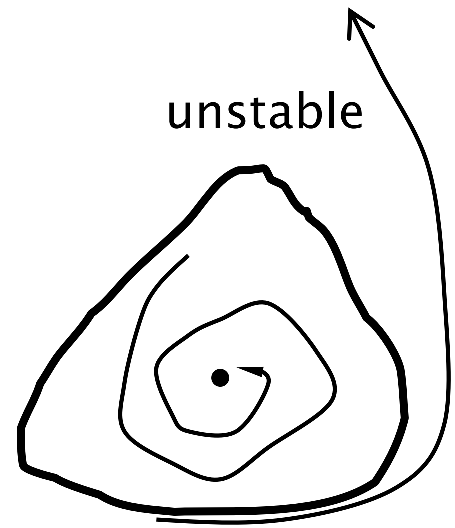
Limit cycle can be:

1. stable
2. unstable
3. half-stable

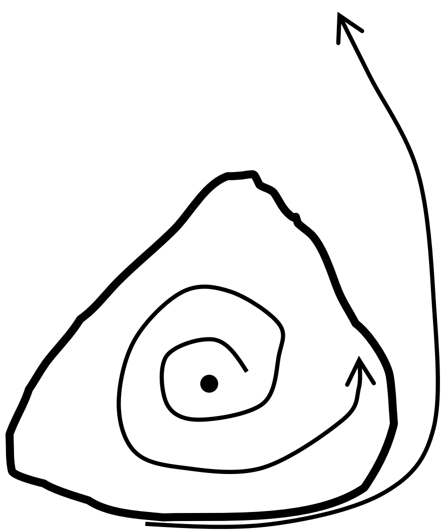
stable

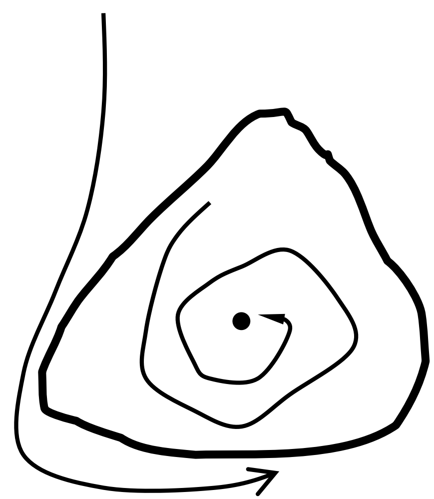


unstable



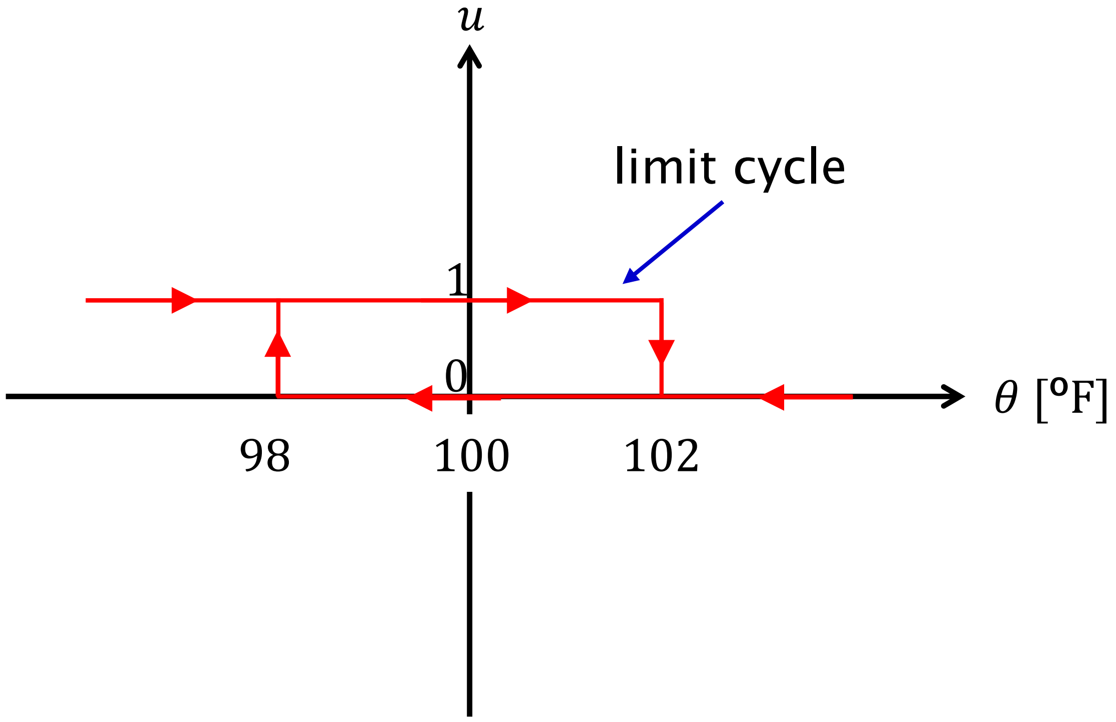
half-stable



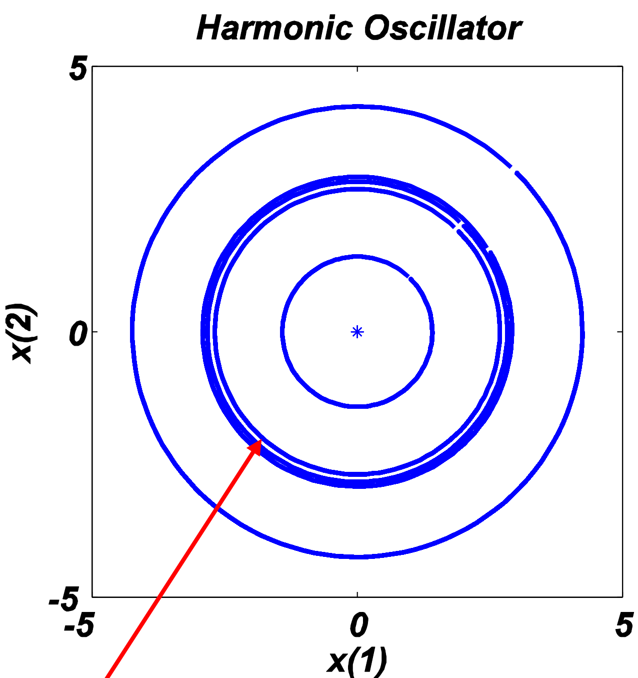


Limit cycle is the hallmark of nonlinear oscillations.

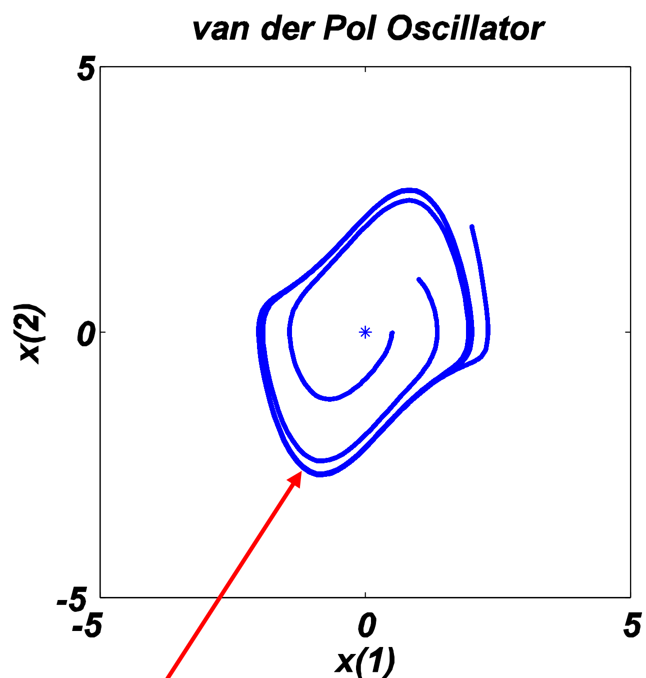
Here is a simple example. Water is heated to 100ºF, and a control system is to maintain this temperature. However the heating system is not perfectly sensitive, i.e. it turns on $(u = 1)$ only when temperature drops to 98ºF, and it shuts off $(u = 0)$ when temperature reaches 102ºF. Assuming the outside temperature is <98ºF, this system will evolve according to the limit cycle.



Note: Not every closed orbit is a limit cycle.



closed but not isolated
(not a limit cycle)



closed and isolated (limit cycle)

Homoclinic orbit: $\ddot{y}(t) = y(t) - y^3(t)$

State variables: $x_1 = y; x_2 = \dot{y}$

State-space form:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_1 - x_1^3$$

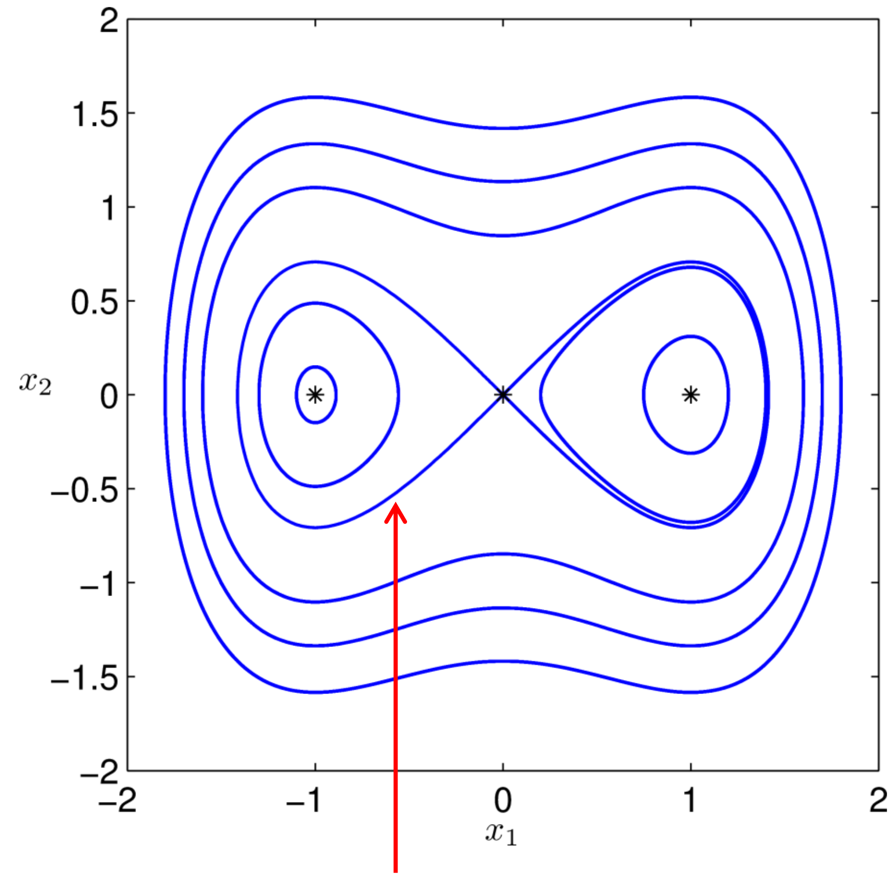
Equilibria:

$$x_1^* = \begin{bmatrix}0\\0\end{bmatrix}, x_2^* = \begin{bmatrix}1\\0\end{bmatrix}, x_3^* = \begin{bmatrix}-1\\0\end{bmatrix}$$

Linearization: $A = \begin{bmatrix}0 & 1\\ 1-3x_1^2 & 0\end{bmatrix}$

$$A_1 = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \lambda_{1,2} = \pm 1, v_1 = \begin{bmatrix}1\\1\end{bmatrix}, v_2 = \begin{bmatrix}1\\-1\end{bmatrix}$$

`homoclinic_orbit.m`



homoclinic orbit:

$$\lim_{t\to\infty} x(t;x_0) = \lim_{t\to-\infty} x(t;x_0)$$

Movement of a ball in a double (potential) energy well:

Potential:

$$V(\xi) = -\frac{g}{2}\xi^2 + \frac{g}{4}\xi^4$$

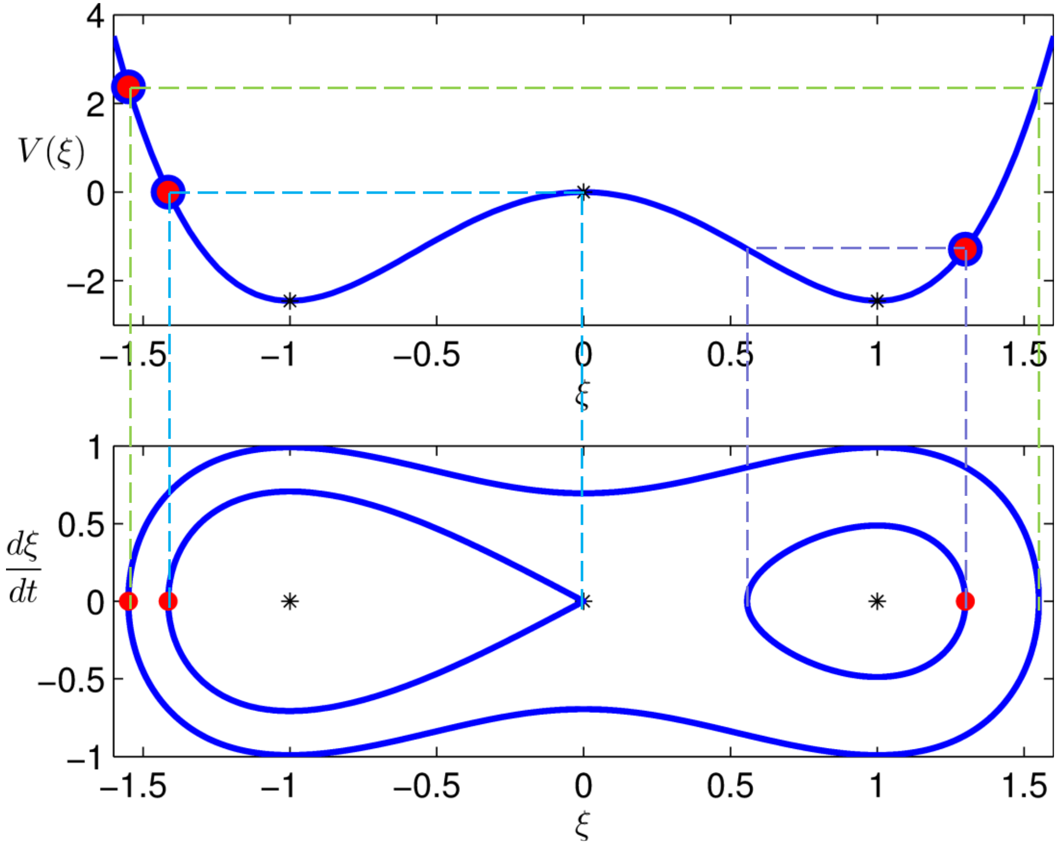
$$E_k + E_p = const. \quad (\text{no frict.})$$

$$\frac{m}{2}\dot{\xi}^2 + mV(\xi) = const.$$

$$2\frac{m}{2}\dot{\xi}\ddot{\xi} + m\frac{\partial V(\xi)}{\partial \xi}\dot{\xi} = 0$$

$$\dot{\xi}\ddot{\xi} + [-g\xi + g\xi^3]\dot{\xi} = 0$$

Either: $\dot{\xi} = 0, (\text{equilibrium})$

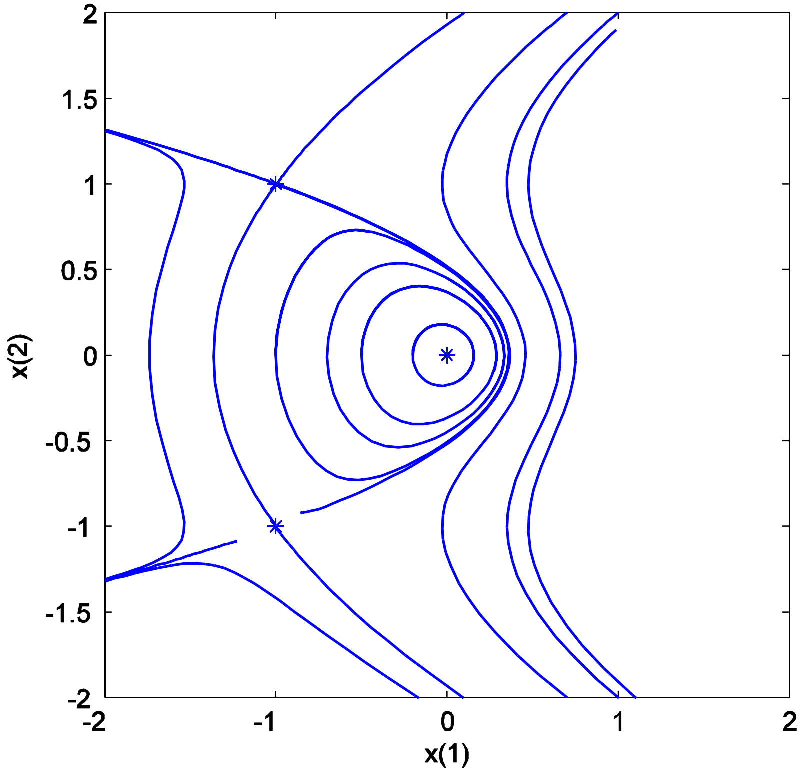


or: $\ddot{\xi} = g\xi - g\xi^3$ (very similar to the original system, the same if $g = 1$)

If you feel adventurous, show that: $x_2 \approx \pm\sqrt{x_1^2 - \frac{x_1^4}{2}}$ is the eq. of the homoclinic orbit.

Heteroclinic orbit:

$$\begin{aligned}\dot{x}_1(t) &= x_2(t) - x_2^3(t)\\ \dot{x}_2(t) &= -x_1(t) - x_2^2(t)\end{aligned}$$



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