

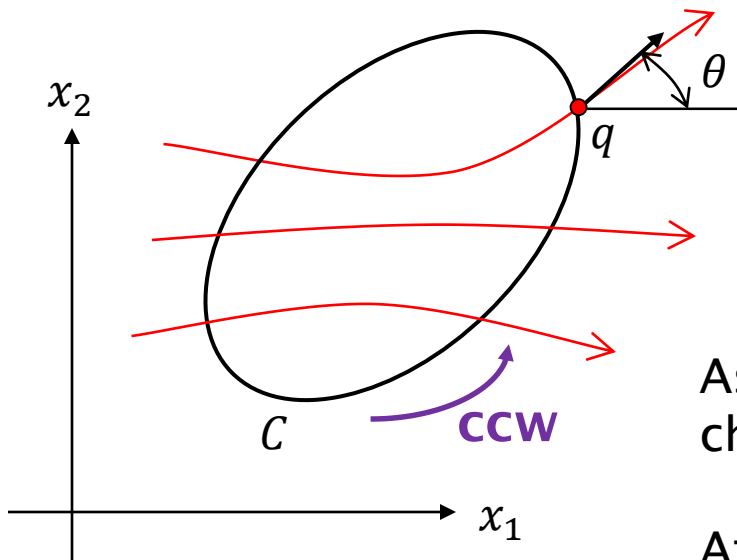
Elementary Index Theory

Second order ($n = 2$) nonlinear time-invariant dynamic system:

$$\frac{dx(t)}{dt} = f(x(t))$$

$f = [f_1, f_2]$ - 2-D vector field. Assume f is smooth.

C - a simple closed curve in $\mathbb{R}^2$ that does not pass through any equilibrium points. C is not necessarily a trajectory.



At each point q on the curve define: $\theta = \theta(q)$ - the angle of the slope of the vector field at point q :

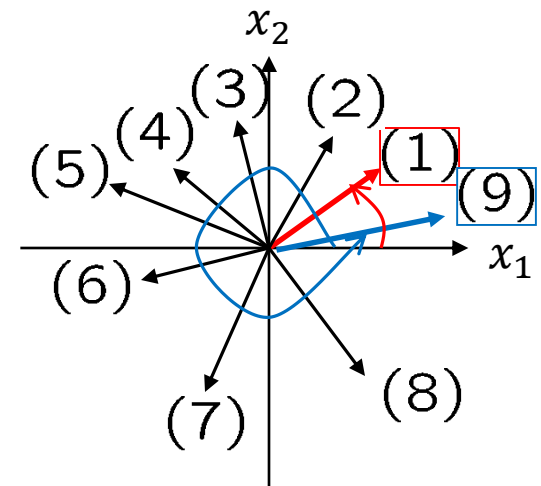
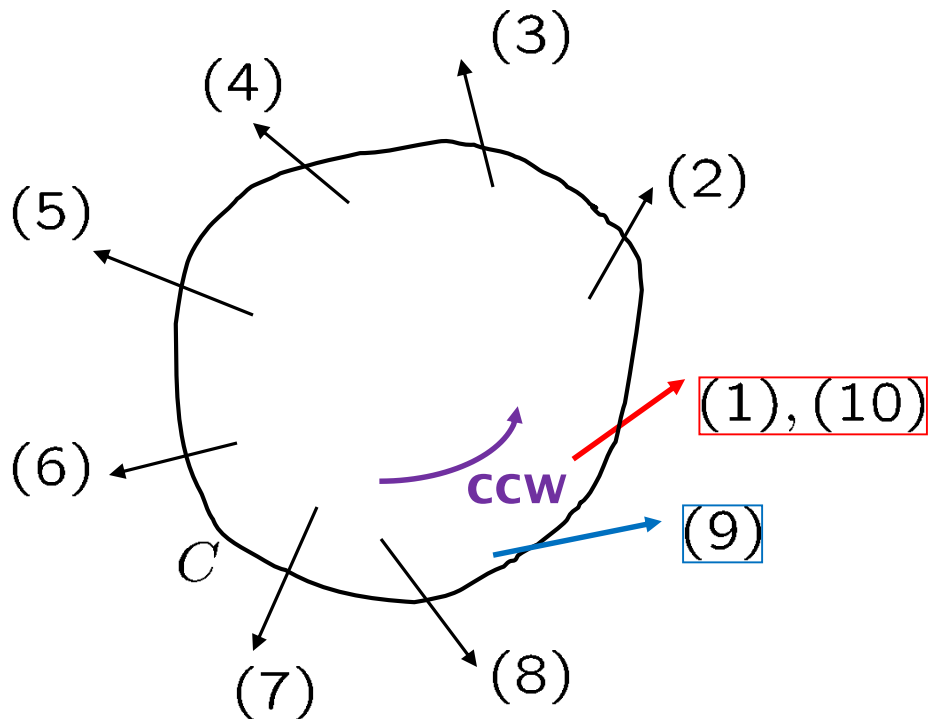
$$\theta(q) := \tan^{-1} \left. \frac{\dot{x}_2}{\dot{x}_1} \right|_{x=q}$$

As q moves CCW around C , the angle θ changes continuously (why?)

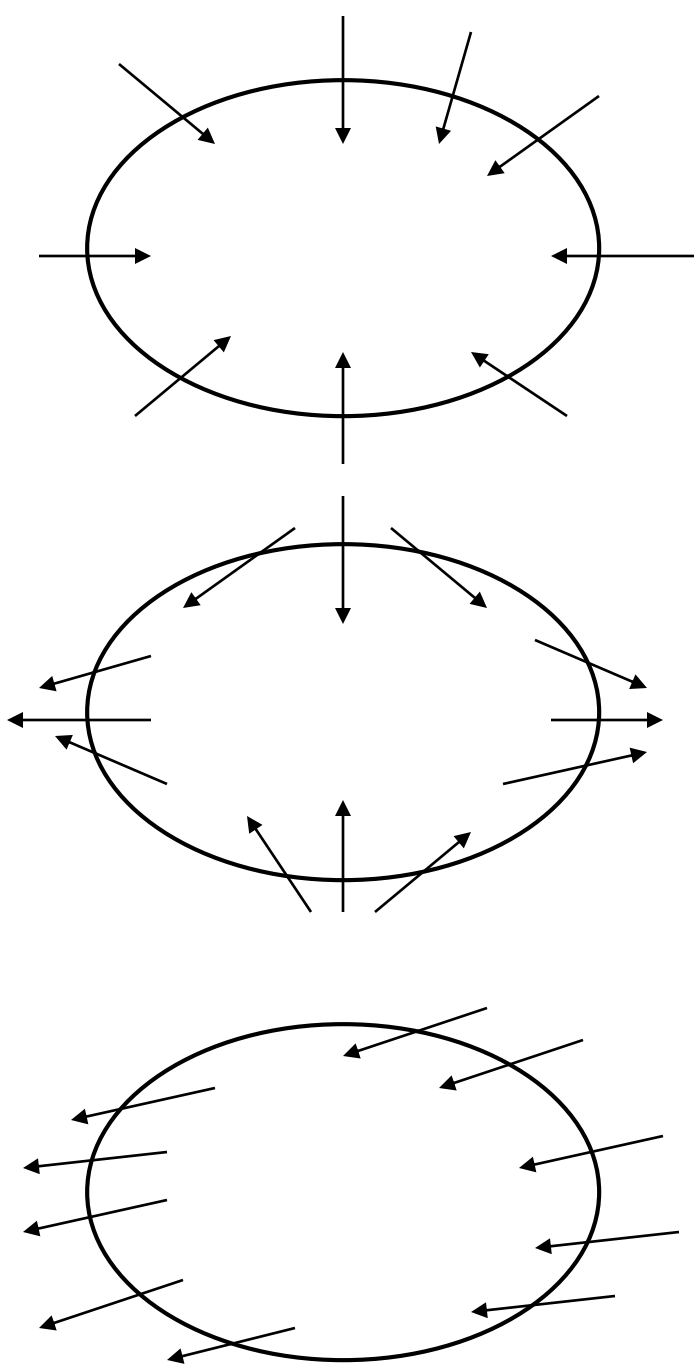
After one revolution, q returns to the initial point q_0 where $\theta = \theta(q_0)$.

If $[\theta]$ is the net change of θ over one CCW revolution, the index of C is defined as:

$$I_C := \frac{1}{2\pi} [\theta]$$



$$I_C = \frac{1}{2\pi} 2\pi = 1$$

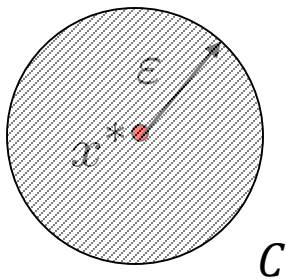


Properties of the Index

- 1) If C does not encircle an equilibrium point, then $I_C = 0$.
- 2) Let C_1 and C_2 be simple closed curves which do not pass through any equilibria. Suppose that C_2 lies entirely in C_1 and that there are no equilibria in between. Then $I_{C_1} = I_{C_2}$.

Index of a point

Suppose x^* is an isolated equilibrium point. The index of x^* is the index of a circle $C := \{x: \|x - x^*\| = \varepsilon\}$, where ε is chosen so small that no other equilibria lie in $\|x - x^*\| \leq \varepsilon$.



$$\|x - x^*\| = \varepsilon \Leftrightarrow \underbrace{\sqrt{(x_1 - x_1^*)^2 + (x_2 - x_2^*)^2}}_{\text{circle with radius } \varepsilon} = \varepsilon$$

Theorem: The index of a closed curve C is equal to the sum of the indices of isolated equilibria inside of C .

Theorem: The index of a focus, node, star, center and degenerate node is +1.

Theorem: The index of a saddle point is -1.

Theorem: The index of any closed trajectory (and a periodic orbit) is +1.

Example: Show that the system

$$\begin{aligned}\frac{dx_1}{dt} &= x_1(4 - x_2 - x_1^2) \\ \frac{dx_2}{dt} &= x_2(x_1 - 1)\end{aligned}$$

has no periodic orbit.

Solution:

Equilibrium points:

$$\begin{aligned}x_1(4 - x_2 - x_1^2) &= 0 \\x_2(x_1 - 1) &= 0\end{aligned}$$

$(x_1 = 0 \text{ or } 4 - x_2 - x_1^2 = 0)$ and $(x_2 = 0 \text{ or } x_1 = 1)$

$$x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, x^* = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, x^* = \begin{bmatrix} -2 \\ 0 \end{bmatrix}, x^* = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$Df(x) = \begin{bmatrix} 4 - x_2 - x_1^2 - 2x_1^2 & -x_1 \\ x_2 & x_1 - 1 \end{bmatrix} = \begin{bmatrix} 4 - x_2 - 3x_1^2 & -x_1 \\ x_2 & x_1 - 1 \end{bmatrix}$$

$$Df\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \lambda_1 = -1, \lambda_2 = 4 \text{ (hyperbolic eq., saddle point } I = -1)$$

$$\text{Eigenvectors: } v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

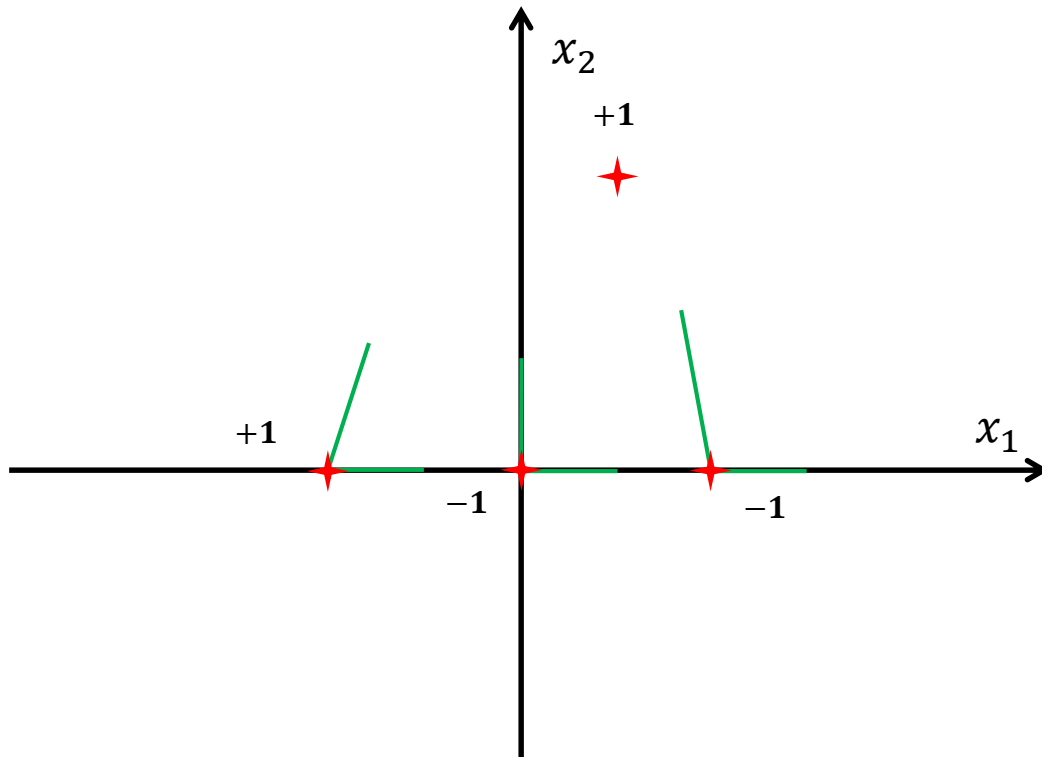
$$Df\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -8 & -2 \\ 0 & 1 \end{bmatrix} \Rightarrow \lambda_1 = -8, \lambda_2 = 1 \text{ (hyperbolic eq., saddle point } I = -1)$$

$$\text{Eigenvectors: } v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ -4.5 \end{bmatrix}$$

$$Df\left(\begin{bmatrix} -2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -8 & 2 \\ 0 & -3 \end{bmatrix} \Rightarrow \lambda_1 = -8, \lambda_2 = -3 \text{ (hyperbolic eq., stable node } I = +1)$$

$$\text{Eigenvectors: } v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 2.5 \end{bmatrix}$$

$$Df\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} -2 & -1 \\ 3 & 0 \end{bmatrix} \Rightarrow \lambda_{1,2} = -1 \pm j\sqrt{2}, \text{ (hyperbolic eq., stable focus } I = +1)$$



Check calculations with `rule_out_closed_orbit.m`
 Check phase portrait with `no_orbit_pp.m`