Gradient (Potential) Systems

$$\frac{dx(t)}{dt} = f\big(x(t)\big) \quad x \in \mathbb{R}^n$$

If there exists a continuously differentiable function $V: \mathbb{R}^n \to \mathbb{R}$, such that $f(x) = -\nabla V(x)$, then the system above is called a gradient system with the potential V.

Note: $V(x)$ is a real-valued scalar function.

What does $f(x) = -\nabla V(x)$ mean?

$$\begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix} = \boxed{f(x) = -\nabla V(x)} = -\underbrace{\begin{bmatrix} \dfrac{\partial V(x)}{\partial x_1} \\ \dfrac{\partial V(x)}{\partial x_2} \\ \vdots \\ \dfrac{\partial V(x)}{\partial x_n} \end{bmatrix}}_{\text{gradient of } V}$$

Hence, $f_i\big(x(t)\big) = -\frac{\partial V(x(t))}{\partial x_i}, \forall i = 1,2,\cdots,n$

Note: continuous differentiability is indeed necessary. It means that $\partial V/\partial x_i$ are continuous functions, which implies that $f_i(x) = -\partial V/\partial x_i$ are continuous, which in turn guarantees the existence of the solution $x(t; t_0, x(t_0))$.

If the potential $V(x)$ is known, it is trivial to find the corresponding gradient system.

$$\dot{x}(t) = -\nabla V(x)$$

Example: Find a gradient system whose potential is $V(x)$:

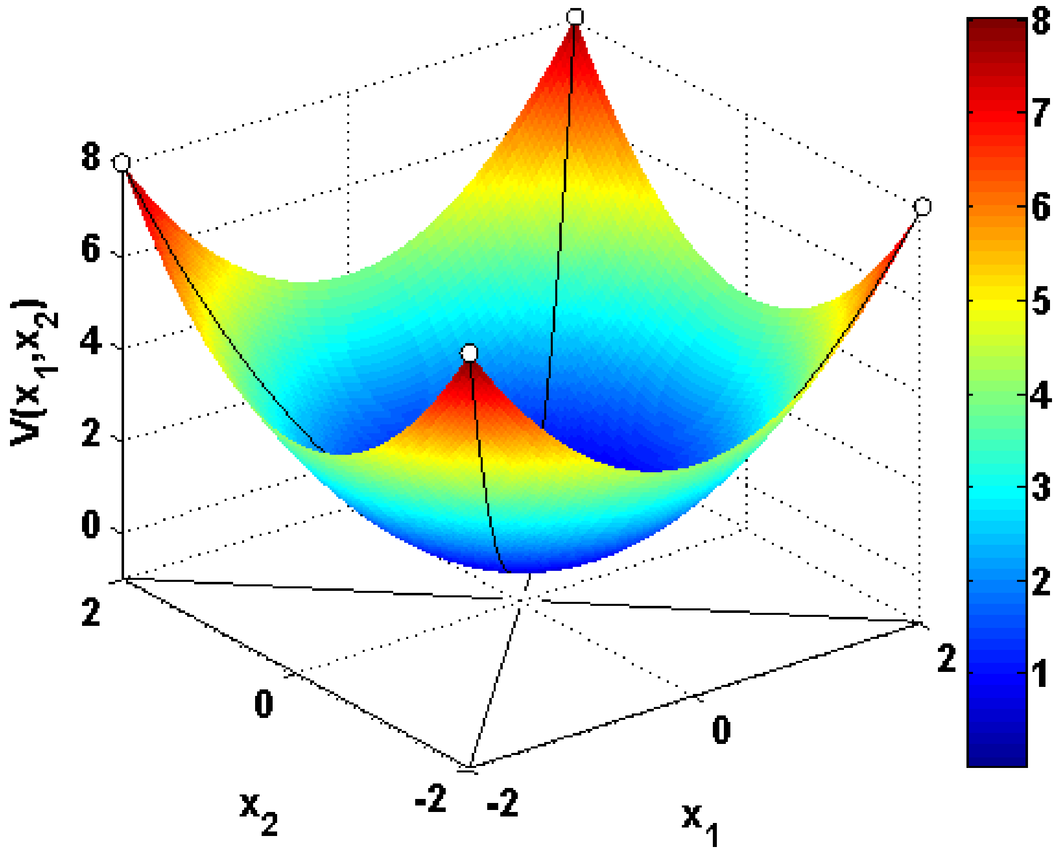
$$\underbrace{V(x)}_{\text{potential}} = x_1^2 + x_2^2$$

$$\frac{\partial V}{\partial x_1} = 2x_1$$

$$\frac{\partial V}{\partial x_2} = 2x_2$$

$$\dot{x}_1 = -\frac{\partial V}{\partial x_1} = -2x_1$$

$$\dot{x}_2 = -\frac{\partial V}{\partial x_2} = -2x_2$$



Conclusion: $\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} x(t)$. Note that we obtained a linear system since $V(x)$ was a 2nd order polynomial.

Example: Find a gradient system whose potential is $V(x)$:

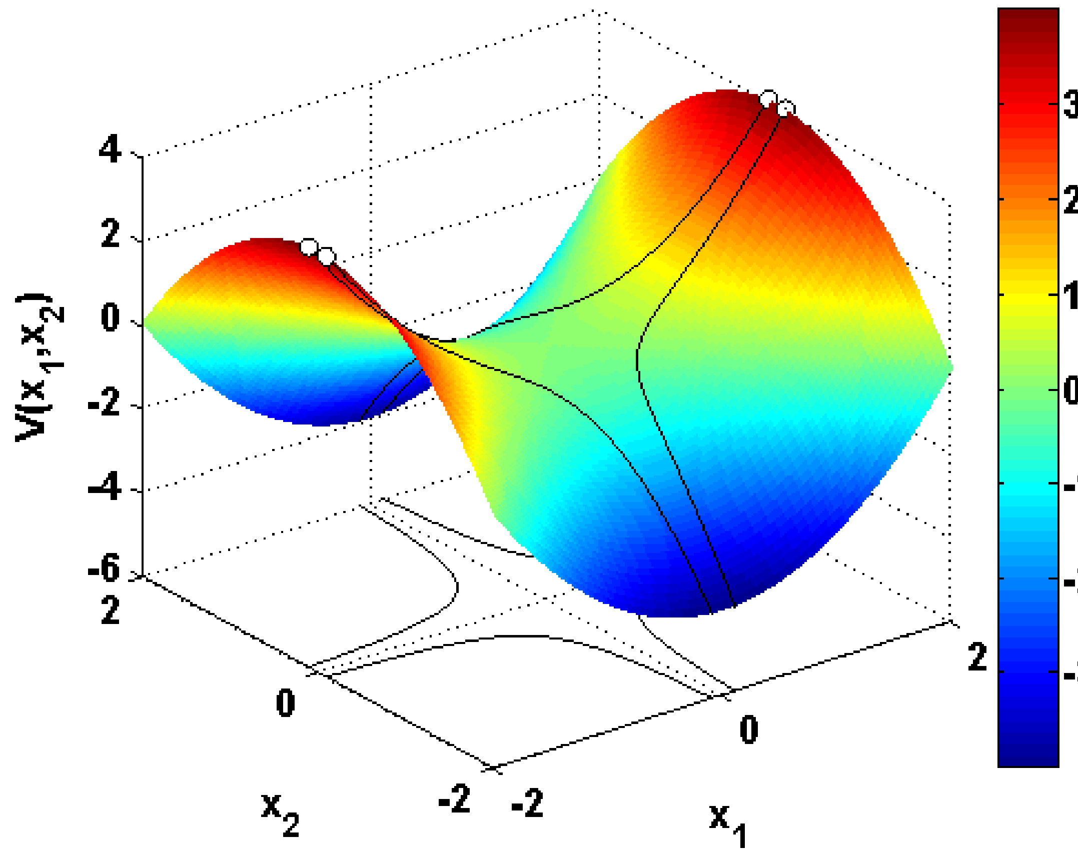
$$\underbrace{V(x)}_{\text{potential}} = x_1^2 - x_2^2$$

$$\frac{\partial V}{\partial x_1} = 2x_1$$

$$\frac{\partial V}{\partial x_2} = -2x_2$$

$$\dot{x}_1 = -\frac{\partial V}{\partial x_1} = -2x_1$$

$$\dot{x}_2 = -\frac{\partial V}{\partial x_2} = 2x_2$$



Conclusion: $\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} x(t)$. Again, we obtained a linear system since we picked a very simple $V(x)$ (2nd order polynomial). In general, this does not have to be the case.

Example: Consider a general nonlinear time-invariant (NLTI) system:

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

Note, it is time-invariant since f_1 and f_2 do not explicitly depend on t.
If this is a gradient system, then: $\frac{\partial f_1}{\partial x_2} = \frac{\partial f_2}{\partial x_1}$

To see this, note that:

$$f_1 = -\frac{\partial V}{\partial x_1} \Rightarrow \frac{\partial f_1}{\partial x_2} = \frac{\partial}{\partial x_2}\left(-\frac{\partial V}{\partial x_1}\right) = -\frac{\partial^2 V}{\partial x_2 \partial x_1}$$
$$f_2 = -\frac{\partial V}{\partial x_2} \Rightarrow \frac{\partial f_2}{\partial x_1} = \frac{\partial}{\partial x_1}\left(-\frac{\partial V}{\partial x_2}\right) = -\frac{\partial^2 V}{\partial x_1 \partial x_2}$$

As long as second partial derivatives are continuous, we have that:

$$\frac{\partial^2 V}{\partial x_1 \partial x_2} = \frac{\partial^2 V}{\partial x_2 \partial x_1}$$

(symmetry of 2^{nd} derivatives)

How to find V if we suspect our system is a gradient system? Example:

$$\dot{x}_1 = x_2^2 + x_2\cos(x_1) = f_1(x_1, x_2)$$
$$\dot{x}_2 = 2x_1x_2 + \sin(x_1) = f_2(x_1, x_2)$$

Since: $f_1 = -\frac{\partial V}{\partial x_1}$, we have:

$$V(x_1, x_2) = -\int f_1(x_1, \underbrace{x_2}_{const})dx_1 = -\int (x_2^2 + x_2\cos(x_1))dx_1$$
$$= -x_2^2x_1 - x_2\sin(x_1) + \phi(x_2)$$

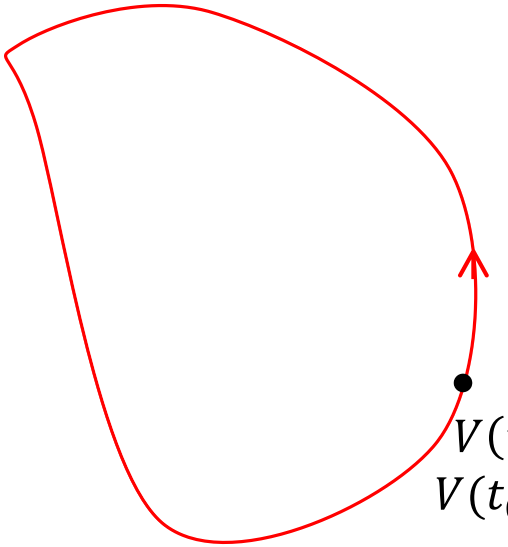
Similarly: $f_2 = -\frac{\partial V}{\partial x_2}$, implies:

$$V(x_1, x_2) = -\int f_2(\underbrace{x_1}_{const}, x_2)dx_1 = -\int (2x_1x_2 + \sin(x_1))dx_2$$
$$= -2x_1\frac{x_2^2}{2} - x_2\sin(x_1) + \psi(x_1) = -x_1x_2^2 - x_2\sin(x_1) + \psi(x_1)$$

Since we want: $-x_2^2x_1 - x_2\sin(x_1) + \phi(x_2) = -x_1x_2^2 - x_2\sin(x_1) + \psi(x_1)$ for all x_1 and x_2, we must have $\phi(x_2) = \psi(x_1) = Const$. Therefore any function of the form: $V(x_1, x_2) = -x_1x_2^2 - x_2\sin(x_1) + Const$. is the potential of the above system.

Theorem: Gradient systems cannot have closed orbits (including limit cycles). In other words they cannot oscillate.

Proof by contradiction. Suppose there is a closed orbit. Starting from t_0, after a period T, the system would come back to the same state, incurring a zero net change of the potential V:



$$\underbrace{\Delta V}_{0} = \int_{t_0}^{t_0+T} dV(t) = \int_{t_0}^{t_0+T} \frac{dV(t)}{dt} dt$$

$$= \int_{t_0}^{t_0+T} \frac{\partial V(t)}{\partial x} \dot{x}(t) dt = \int_{t_0}^{t_0+T} \underbrace{\nabla^T V(x)}_{-f^T(x)} \dot{x}(t) dt$$

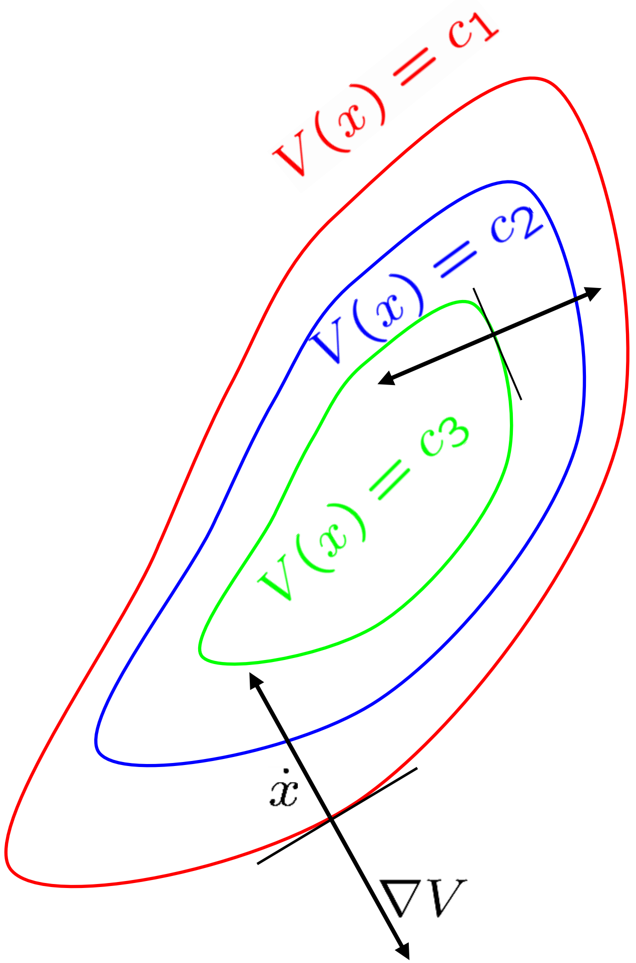
$$= -\int_{t_0}^{t_0+T} \underbrace{f^T(x)}_{\dot{x}^T} \dot{x}(t) dt = \int_{t_0}^{t_0+T} \|\dot{x}(t)\|^2 dt = 0$$

The above is true iff: $\|\dot{x}(t)\| = 0$, which implies that $\underbrace{\dot{x} \equiv 0}_{\text{equilibrium}}$, which contradicts the initial premise that the system evolves along a closed orbit.

The intuition is clear here:

$V(x) = const.$ contours

$c_1 > c_2 > c_3$



Note, since $\nabla V(x)$ is a gradient of V, it is perpendicular to the contour (or more precisely the tangent to the contour).

If this is a gradient system, then dx/dt is perpendicular to the contour since

$$\frac{dx}{dt} = -\nabla V(x)$$

The solutions evolve along trajectories which are perpendicular to the contours

$$V(x) = const.$$

If trajectories are perpendicular to the contours, then no trajectory can be closed (the system evolves by following the gradient).

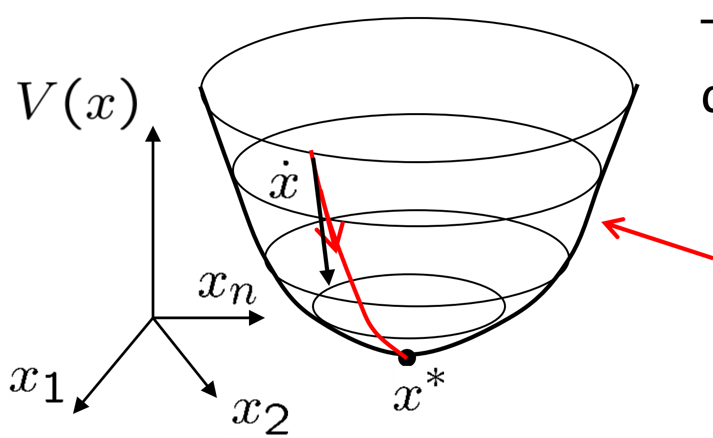
Theorem [Lyapunov 1892]: Let x^* be an equilibrium point of an n-th order system $dx(t)/dt = f\big(x(t)\big)$. If there exists a continuously differentiable function $V: \mathbb{R}^n \to \mathbb{R}$ such that:

1) $V(x) > 0$ for all $x \neq x^*$ and $V(x^*) = 0$.
2) $dV(x)/dt \leq 0$ for all $x \neq x^*$

Then x^* is Lyapunov stable.

If 2) $dV(x)/dt < 0$ for all $x \neq x^*$, then x^* is globally asymptotically stable (Lyapunov stable + globally attractive). Such a function V is called the Lyapunov function.

If the Lyapunov function exists, the system cannot have a closed (or periodic) orbit.



The trajectories move monotonically downhill (toward x^*), because $dV/dt < 0$

Generalized energy function that's always dissipated, except at $x = x^*$

Example: show that the system:

$$\dot{x}_1 = x_2 - x_1^3$$
$$\dot{x}_2 = -x_1 - x_2^3$$

has no closed orbits.

Hint: $V(x) = ax_1^2 + bx_2^2$

Find equilibria:

$$\dot{x}_1 = 0 \Rightarrow x_2 = x_1^3$$
$$\dot{x}_2 = 0 \Rightarrow x_1 = -x_2^3 \Rightarrow x_1^3 = -x_2^9 \Rightarrow x_2 + x_2^9 = 0 \Rightarrow \boxed{x_2 = 0}, \boxed{x_1 = 0}$$
$$x^\star = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Check the conditions of the Lyapunov Theorem:

1) $V(x) > 0, \forall x \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (as long as $a > 0$ and $b > 0$), and $V(x^\star) = 0$

2) $\frac{dV(x)}{dt} = \frac{\partial V(x)}{\partial x} \dot{x}(t) = \begin{bmatrix} 2ax_1 & 2bx_2 \end{bmatrix} \begin{bmatrix} x_2 - x_1^3 \\ -x_1 - x_2^3 \end{bmatrix}$

$$= 2ax_1x_2 - 2ax_1^4 - 2bx_1x_2 - 2bx_2^4 \quad (a = b)$$
$$= -2a \underbrace{\left(x_1^4 + x_2^4\right)}_{>0} < 0 \text{ as long as } x \neq x^\star$$

Conclusion: $V(x) = a(x_1^2 + x_2^2)$ is a Lyapunov function, therefore the system cannot have closed orbits.

Example: show that the system:

$$\dot{x}_1 = -x_1 + 4x_2$$
$$\dot{x}_2 = -x_1 - x_2^3$$

has no closed orbits.

Hint: $V(x) = x_1^2 + bx_2^2$

Find equilibria:

$$\dot{x}_1 = 0 \Rightarrow x_1 = 4x_2$$
$$\dot{x}_2 = 0 \Rightarrow x_1 = -x_2^3 \Rightarrow x_2^3 + 4x_2 = 0 \Rightarrow \boxed{x_2 = 0}, \boxed{x_1 = 0}$$
$$x^\star = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Check the conditions of the Lyapunov Theorem:

1) $V(x) > 0, \forall x \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (as long as $b > 0$), and $V(x^\star) = 0$

2) $\frac{dV(x)}{dt} = \frac{\partial V(x)}{\partial x}\ \dot{x}(t) = [2x_1 \quad 2bx_2]\begin{bmatrix} -x_1 + 4x_2 \\ -x_1 - x_2^3 \end{bmatrix}$

$$= -2x_1^2 + 8x_1x_2 - 2bx_1x_2 - 2bx_2^4 \ \ (b = 4)$$
$$= -2\underbrace{(x_1^2 + 4x_2^4)}_{>0} < 0 \ \text{ as long as } x \neq x^\star$$

Conclusion: $V(x) = x_1^2 + 4x_2^2$ is a Lyapunov function, therefore the system cannot have closed orbits.

Conservative Systems

$$dx(t)/dt = f\big(x(t)\big) \qquad x \in \mathbb{R}^n$$

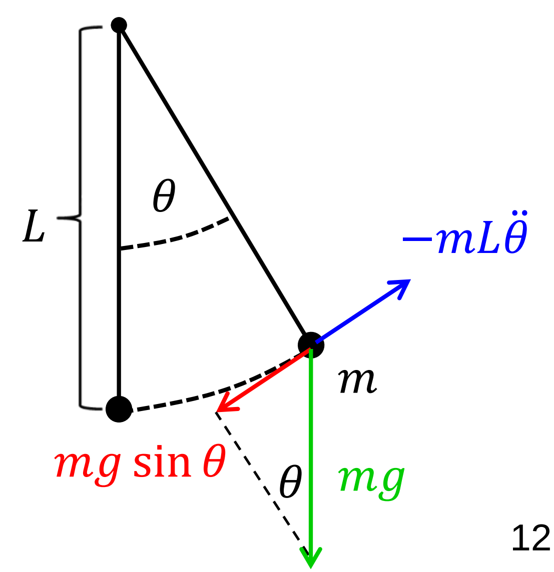
If there exists a continuous function $E: \mathbb{R}^n \to \mathbb{R}$, that is:
(i) non-constant on every open set and
(ii) constant along every trajectory, i.e. $dE(x)/dt = 0$

then the system above is called a <u>conservative system</u>, and the quantity $E(x)$ represents a conserved quantity.

Conservative systems <u>have no attracting equilibria</u>. For conservative systems, the linearized center is also a nonlinear center (despite being non-hyperbolic).

Example: pendulum with no friction:

$$\underbrace{mL\ddot{\theta}}_{inertia} + \underbrace{mg\sin\theta}_{gravity} = 0$$



State variables:

$$x_1 \coloneqq \theta \Rightarrow \dot{x}_1 = x_2$$
$$x_2 \coloneqq \dot{\theta} \Rightarrow \dot{x}_2 = -\frac{g}{L}\sin x_1$$

The conserved quantity candidate—energy function:

$$E = E_k + E_p = \frac{1}{2}mv^2 + mgh$$

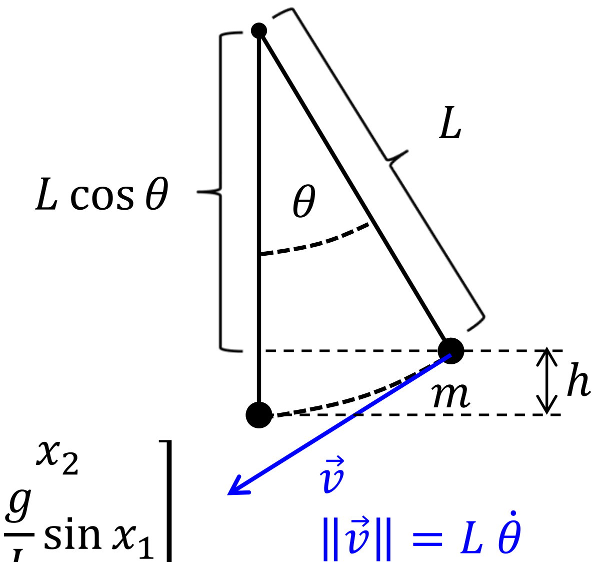
$$E = \frac{1}{2}m(L\dot{\theta})^2 + mg(L - L\cos\theta)$$

$$E \propto \frac{1}{2}L^2\left(\dot{\theta}\right)^2 + gL(1-\cos\theta)$$

$$E \propto \frac{L}{2}x_2^2 + g(1-\cos x_1)$$

$$\frac{dE(x)}{dt} = \frac{\partial E}{\partial x}\,\dot{x}(t) = [g\sin x_1 \quad Lx_2]\begin{bmatrix} x_2 \\ -\frac{g}{L}\sin x_1 \end{bmatrix}$$

$$\boxed{\frac{dE(x)}{dt}} = gx_2\sin x_1 - gx_2\sin x_1 = \boxed{0}$$



Since $E(x) = Const.$, this system cannot have attracting equilibria.