Poincaré-Bendixson Theory

$$\frac{dx(t)}{dt} = f\big(x(t)\big) \quad x \in \mathbb{R}^2 \quad \text{(2nd order system)}$$

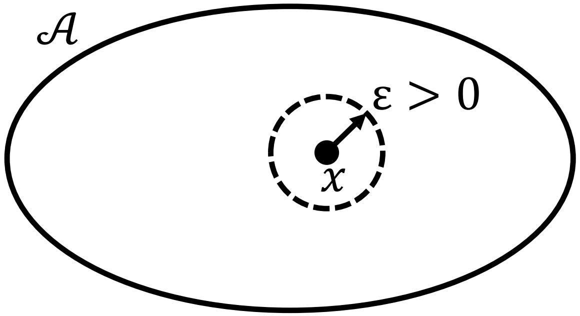
Assume $f(x)$ is continuously differentiable throughout this lecture.

Historical perspective:

1. Poincaré [late 19th century]
2. Bendixson [1901] – more rigorous result

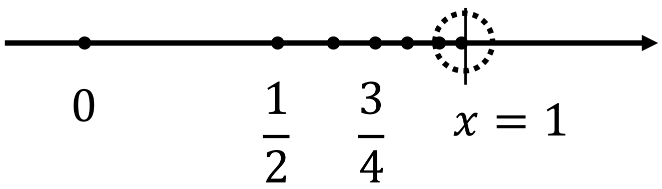
Theorem [Poincaré]: Let $\gamma := \{x(t; x_0), t \geq 0\}$ be a bounded (semi)orbit, and let Ω be its limit set (set of all accumulation points of $x(t; x_0)$ as $t \to \infty$). Then, Ω either contains an equilibrium point or Ω is a periodic orbit.

Accumulation point: If every open neighborhood $B(x,\varepsilon)$ of x contains infinitely many points of $\mathcal{A}$, then x is called an accumulation point of $\mathcal{A}$.



Note: x does not need to belong to $\mathcal{A}$.
Example:

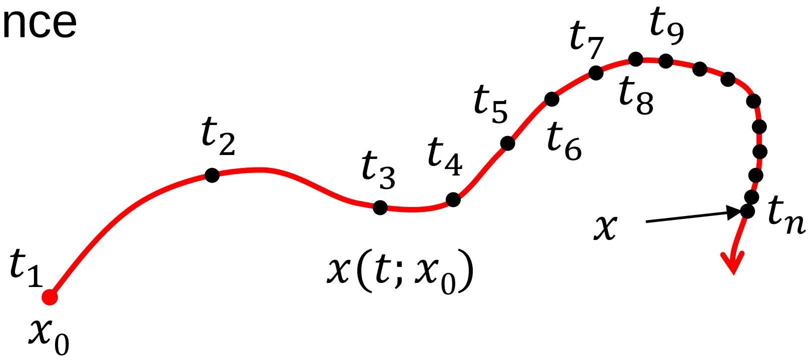
$$\mathcal{A} = \left\{1 - \frac{1}{n}, n \in \mathbb{N}\right\} = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots\right\}$$



Thus, $x = \lim_{n\to\infty}\left(1 - \frac{1}{n}\right) = 1$ is an accumulation point of $\mathcal{A}$, but $x \notin \mathcal{A}$.

For an orbit $\gamma := \{x(t;x_0), t \geq 0\}$, we call a point x an accumulation point if there exists a sequence $\{t_n, n \in \mathbb{N}\}$ such that

$$\lim_{n\to\infty} t_n = \infty$$
$$\lim_{n\to\infty} x(t_n;x_0) = x$$

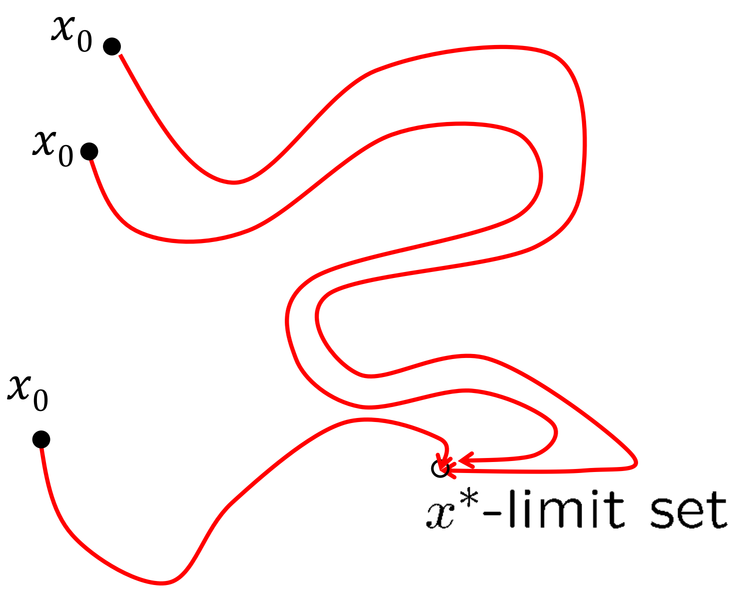


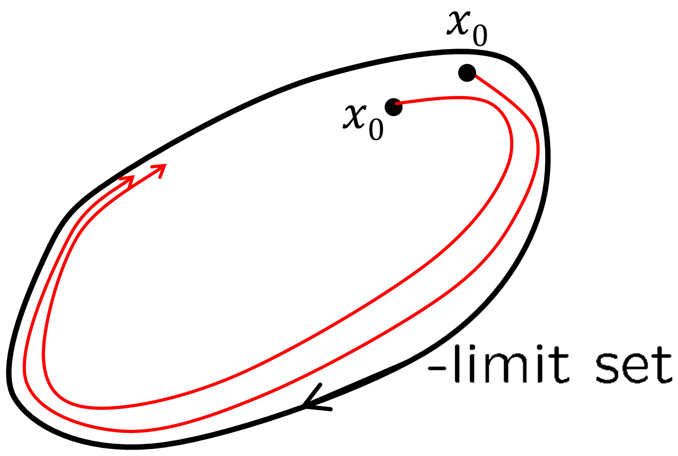
The theorem is a lot simpler than it appears.

Basically, 3 things can happen to a trajectory in $\mathbb{R}^2$

1. it diverges (explodes out of bounds)
2. it approaches an equilibrium point
3. it approaches a periodic orbit

$\left.\begin{array}{l} 2. \\ 3. \end{array}\right\}$ Poincaré Theorem





Theorem [Bendixson]: Suppose $\mathcal{D} \subseteq \mathbb{R}^2$ is simply connected and $div\, f := \partial f_1/\partial x_1 + \partial f_2/\partial x_2$ has a constant (+/-) sign on $\mathcal{D}$. Then there does not exist a periodic orbit entirely lying in $\mathcal{D}$.

Example: investigate whether the system

$$dx_1/dt = -x_2 - x_1({x_1}^2 + {x_2}^2 - 1)$$
$$dx_2/dt = x_1 - x_2({x_1}^2 + {x_2}^2 - 1)$$

has a periodic orbit in the region $\mathcal{D}$

(A) $\mathcal{D} = \left\{x \in \mathbb{R}^2\colon\ \|x\| < \frac{1}{\sqrt{2}}\right\}$

(B) $\mathcal{D} = \{x \in \mathbb{R}^2\colon\ \|x\| < 2\}$

Note: f_1 and f_2 are continuously differentiable.

$$\boxed{div\, f} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = -(x_1^2 + x_2^2 - 1) - 2x_1^2 - (x_1^2 + x_2^2 - 1) - 2x_2^2 = \boxed{-4\left(x_1^2 + x_2^2 - \frac{1}{2}\right)}$$

(A) $\mathcal{D} = \left\{x \in \mathbb{R}^2 \colon \ \|x\| < \frac{1}{\sqrt{2}}\right\} = \left\{x \in \mathbb{R}^2 \colon \sqrt{x_1^2 + x_2^2} < \frac{1}{\sqrt{2}}\right\} = \left\{x \in \mathbb{R}^2 \colon x_1^2 + x_2^2 < \frac{1}{2}\right\}$

Thus, on $\mathcal{D}$ we have: $x_1^2 + x_2^2 - \frac{1}{2} < 0$. Since $div\, f = -4\left(x_1^2 + x_2^2 - \frac{1}{2}\right)$, on $\mathcal{D}$ we have: $div\, f > 0$

Conclusion: There is no periodic orbit entirely lying in $\mathcal{D}$.

(B) $\mathcal{D} = \{x \in \mathbb{R}^2 \colon \|x\| < 2\} = \left\{x \in \mathbb{R}^2 \colon \sqrt{x_1^2 + x_2^2} < 4\right\}$

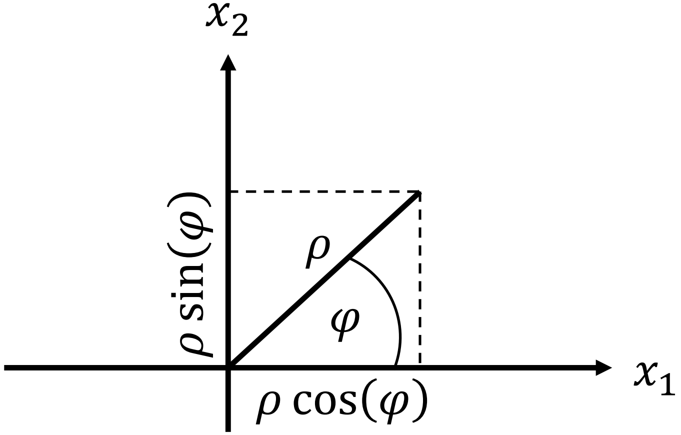
Since $div\, f = -4\left(x_1^2 + x_2^2 - \frac{1}{2}\right)$, $div\, f$ changes the sign when $x_1^2 + x_2^2 = \frac{1}{2}$ or $\sqrt{x_1^2 + x_2^2} = \frac{1}{\sqrt{2}}$.

Conclusion: $div\, f$ changes the sign on $\mathcal{D}$, therefore, there could be a periodic orbit entirely lying in $\mathcal{D}$.

To test this hypothesis, we switch to polar coordinates.

$$dx_1/dt = -x_2 - x_1({x_1}^2 + {x_2}^2 - 1)$$
$$dx_2/dt = x_1 - x_2({x_1}^2 + {x_2}^2 - 1)$$

Polar coordinates: $x_1 = \rho\cos(\varphi), x_2 = \rho\sin(\varphi)$



$$\dot{x}_1 = \dot{\rho}\cos(\varphi) - \rho\sin(\varphi)\dot{\varphi}$$
$$\dot{x}_2 = \dot{\rho}\sin(\varphi) + \rho\cos(\varphi)\dot{\varphi}$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}\cos(\varphi) & -\rho\sin(\varphi)\\ \sin(\varphi) & \rho\cos(\varphi)\end{bmatrix}\begin{bmatrix}\dot{\rho}\\ \dot{\varphi}\end{bmatrix}$$

$$\begin{bmatrix}\dot{\rho}\\ \dot{\varphi}\end{bmatrix} = \begin{bmatrix}\cos(\varphi) & -\rho\sin(\varphi)\\ \sin(\varphi) & \rho\cos(\varphi)\end{bmatrix}^{-1}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}$$

$$\begin{bmatrix}\dot{\rho}\\ \dot{\varphi}\end{bmatrix} = \frac{1}{\rho\cos^2(\varphi)+\rho\sin^2(\varphi)}\begin{bmatrix}\rho\cos(\varphi) & \rho\sin(\varphi)\\ -\sin(\varphi) & \cos(\varphi)\end{bmatrix}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}$$

$$\begin{bmatrix}\dot{\rho}\\ \dot{\varphi}\end{bmatrix} = \frac{1}{\rho}\begin{bmatrix}\rho\cos(\varphi) & \rho\sin(\varphi)\\ -\sin(\varphi) & \cos(\varphi)\end{bmatrix}\begin{bmatrix}-\rho\sin(\varphi) - \rho\cos(\varphi)\,(\rho^2\cos^2(\varphi) + \rho^2\sin^2(\varphi) - 1)\\ \rho\cos(\varphi) - \rho\sin(\varphi)\,(\rho^2\cos^2(\varphi) + \rho^2\sin^2(\varphi) - 1)\end{bmatrix}$$

$$\begin{bmatrix}\dot{\rho}\\ \dot{\varphi}\end{bmatrix} = \begin{bmatrix}\rho\cos(\varphi) & \rho\sin(\varphi)\\ -\sin(\varphi) & \cos(\varphi)\end{bmatrix}\begin{bmatrix}-\sin(\varphi) - \cos(\varphi)\,(\rho^2 - 1)\\ \cos(\varphi) - \sin(\varphi)\,(\rho^2 - 1)\end{bmatrix}$$

$$\begin{bmatrix}\dot\rho\\ \dot\varphi\end{bmatrix} = \begin{bmatrix}\rho\cos(\varphi) & \rho\sin(\varphi)\\ -\sin(\varphi) & \cos(\varphi)\end{bmatrix}\begin{bmatrix}-\sin(\varphi)-\cos(\varphi)\,(\rho^2-1)\\ \cos(\varphi)-\sin(\varphi)\,(\rho^2-1)\end{bmatrix}$$

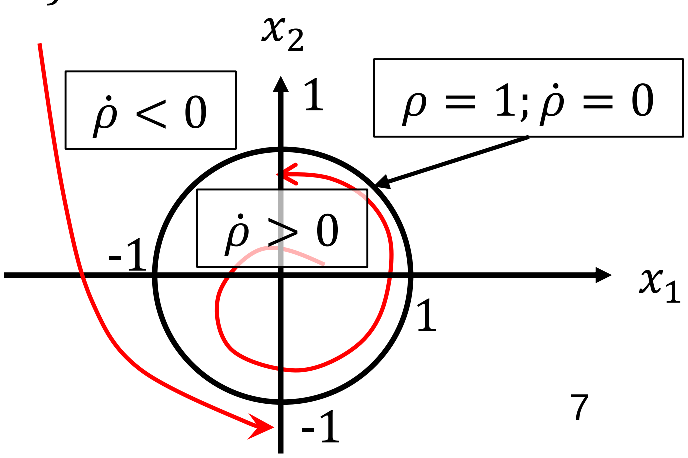
$$\begin{bmatrix}\dot\rho\\ \dot\varphi\end{bmatrix} = \begin{bmatrix}-\rho\cos(\varphi)\sin(\varphi)-\rho\cos^2(\varphi)\,(\rho^2-1)+\rho\sin(\varphi)\cos(\varphi)-\rho\sin^2(\varphi)(\rho^2-1)\\ \sin^2(\varphi)+\sin(\varphi)\cos(\varphi)\,(\rho^2-1)+\cos^2(\varphi)-\sin(\varphi)\cos(\varphi)\,(\rho^2-1)\end{bmatrix}$$

$$\begin{bmatrix}\dot\rho\\ \dot\varphi\end{bmatrix} = \begin{bmatrix}-\rho(\rho^2-1)\\ 1\end{bmatrix}$$

$$\boxed{\begin{matrix}\dot\rho = \rho(1-\rho^2)\\ \dot\varphi = 1\end{matrix}}$$

(A) $\mathcal{D} = \left\{x \in \mathbb{R}^2\colon \|x\| < \frac{1}{\sqrt{2}}\right\} = \left\{\rho \ge 0, \varphi \in [0, 2\pi]\colon \rho < \frac{1}{\sqrt{2}}\right\}$

(B) $\mathcal{D} = \{x \in \mathbb{R}^2\colon \|x\| < 2\} = \{\rho \ge 0, \varphi \in [0, 2\pi]\colon \rho < 2\}$



Theorem [Poincare-Bendixson]: Suppose that:

(1) $\mathcal{D} \subseteq \mathbb{R}^2$ is a closed and bounded set
(2) $dx(t)/dt = f(x(t))$ is a continuously differentiable vector field on an open set $S \supset \mathcal{D}$
(3) There are no equilibrium points in $\mathcal{D}$.
(4) There exists a trajectory that is confined to $\mathcal{D}$, meaning if it starts in $\mathcal{D}$ it remains in $\mathcal{D}$ for all future times.

Then, either the trajectory is a closed orbit or its limit set is a closed orbit.

Example: use Poincaré-Bendixson Theorem to investigate whether the system:

$$\frac{d\rho}{dt} = \rho(1-\rho^2)$$
$$\frac{d\varphi}{dt} = -1$$

has a limit cycle on $\mathcal{D}$, where

(A) $\mathcal{D} = \{\rho \geq 0, \varphi \in [0, 2\pi]: 1-\varepsilon \leq \rho \leq 1+\varepsilon, \qquad 0 < \varepsilon < 1\}$
(B) $\mathcal{D} = \{\rho \geq 0, \varphi \in [0, 2\pi]: 1+\varepsilon \leq \rho \leq 2\}$

1) $\mathcal{D}$ is closed and bounded in both (A) and (B)

2) f_ρ and f_φ are continuously differentiable on any open set $S \supset \mathcal{D}$

3) There are no equilibrium points in $\mathcal{D}$.

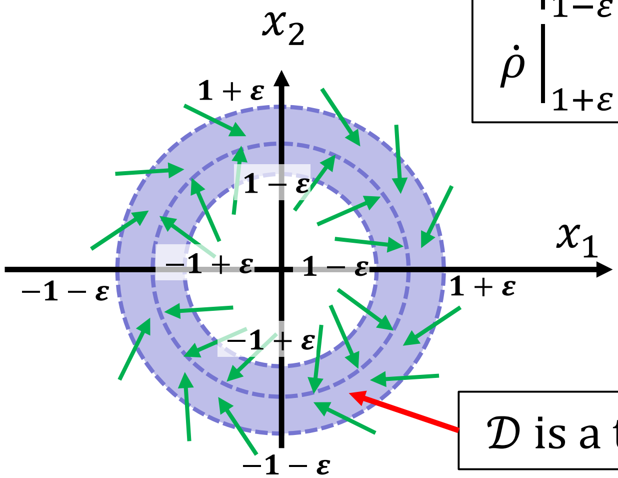
$$\boxed{\begin{aligned}\frac{d\rho}{dt} &= \rho(1-\rho^2)\\ \frac{d\varphi}{dt} &= -1\end{aligned}}$$

Equilibria: $\dot{\rho} = 0 \Rightarrow \rho(1-\rho^2) = 0 \Rightarrow \boxed{\rho^* = 0}$ or $\boxed{\rho^* = 1}$

Since $\dot{\varphi} \neq 0$, only $\rho^* = 0$ is an equilibrium. Note $\rho^* = 0 \Rightarrow \boxed{x_1^* = x_2^* = 0}$

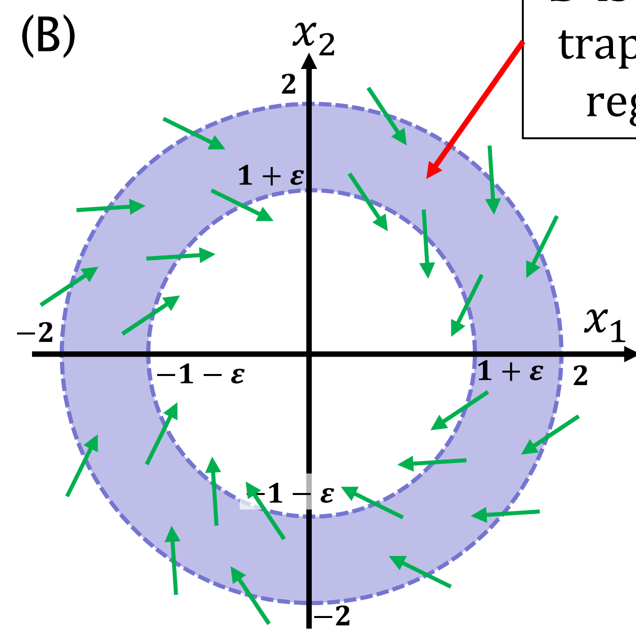
(A)

$$\boxed{\begin{aligned}\dot{\rho}\Big|_{1-\varepsilon} &> 0\\ \dot{\rho}\Big|_{1+\varepsilon} &< 0\end{aligned}}$$



$\mathcal{D}$ is a trapping region

(B)



$\mathcal{D}$ is not a trapping region

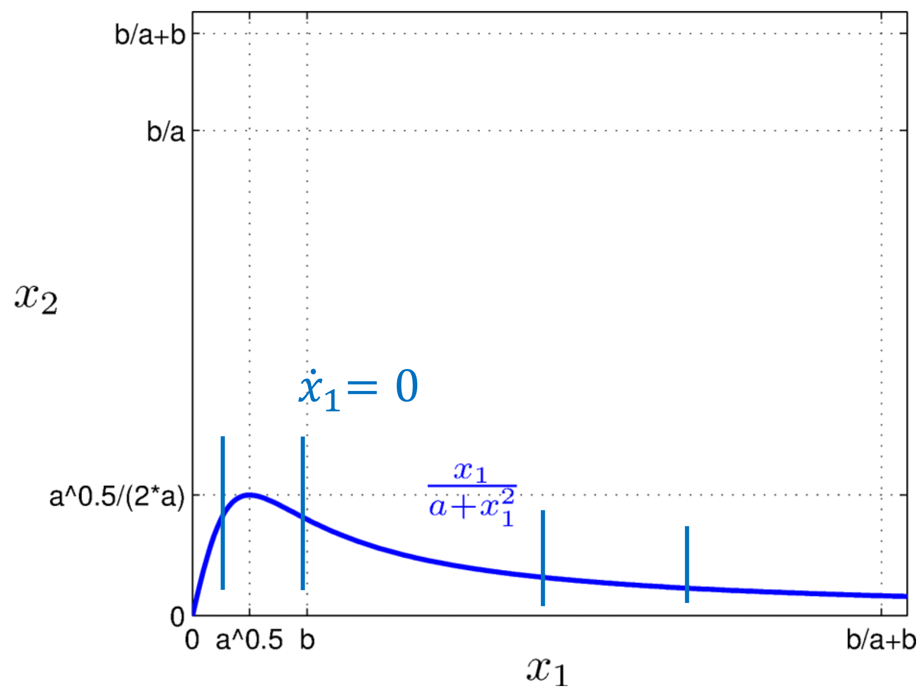
Example: glycolysis equation:

$$\boxed{\begin{aligned} \dot{x}_1 &= -x_1 + a x_2 + x_1^2 x_2 \\ \dot{x}_2 &= b - a x_2 - x_1^2 x_2 \qquad\qquad a, b > 0 \end{aligned}}$$

Note: $x_1, x_2 > 0$ (since they are concentrations)

Nullclines:

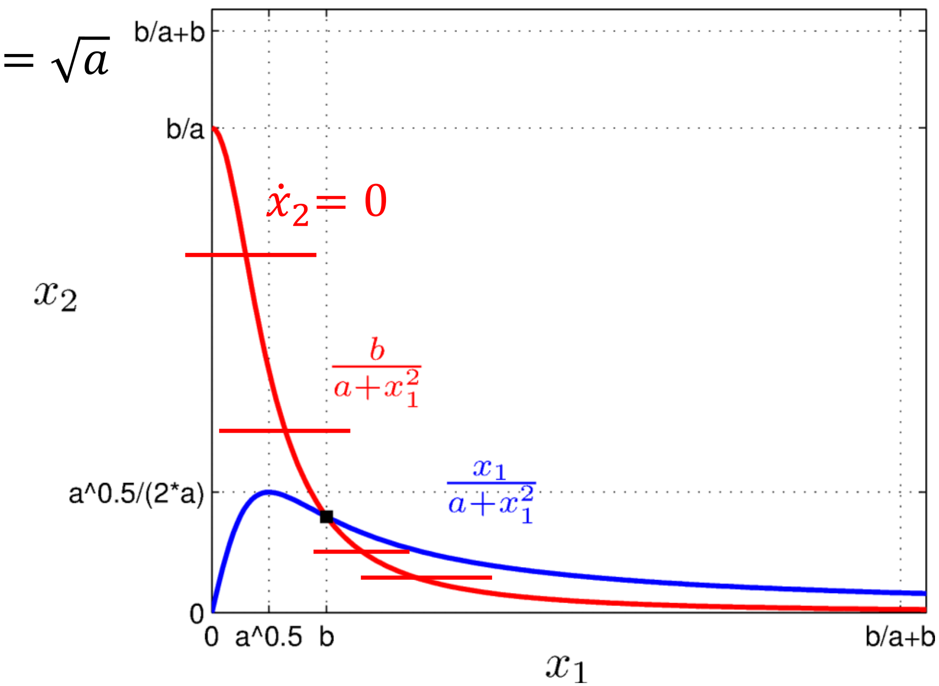
$$\dot{x}_1 = 0$$
$$x_2(a + x_1^2) = x_1$$
$$x_2 = \frac{x_1}{(a + x_1^2)}$$



Peak at: $x_2' = 0 \Rightarrow a + x_1^2 - 2x_1^2 = 0 \Rightarrow x_1 = \sqrt{a}$

Second nullcline:

$$\dot{x}_2 = 0$$
$$x_2(a + x_1^2) = b$$
$$x_2 = \frac{b}{(a + x_1^2)}$$



Peak at: $x_2' = 0 \Rightarrow -2bx_1 = 0 \Rightarrow x_1 = 0$

Equilibrium: $x_1^* = b, x_2^* = \frac{b}{a+b^2}$

How to find the trapping region $\mathcal{D}$?

When $\boldsymbol{x_2 = 0, \dot{x}_1 = -x_1, \dot{x}_2 = b}$

When $\boldsymbol{x_1 = 0, \dot{x}_1 = a x_2, \dot{x}_2 = b - a x_2}$

Thus, it makes sense that $x_1 = 0$ and $x_2 = 0$ are borders of $\mathcal{D}$.

When $\boldsymbol{x_2 = \frac{b}{a}, \dot{x}_2 = -x_1^2 \frac{b}{a}}$

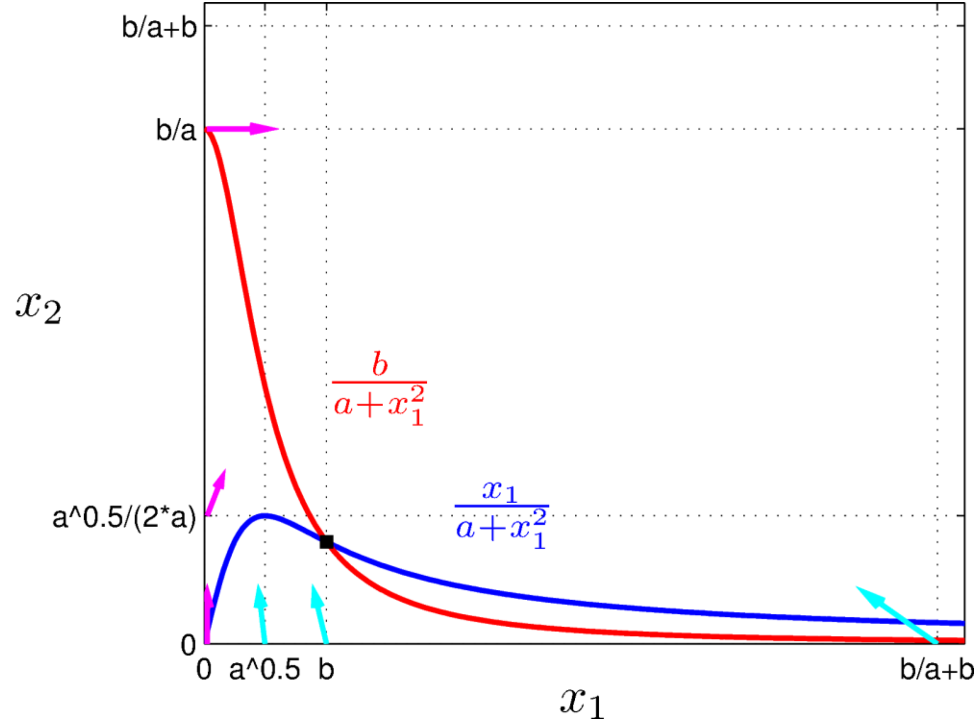
Therefore, $\boldsymbol{\dot{x}_2 < 0}$, and it can also be shown that $\boldsymbol{\dot{x}_1 > 0}$

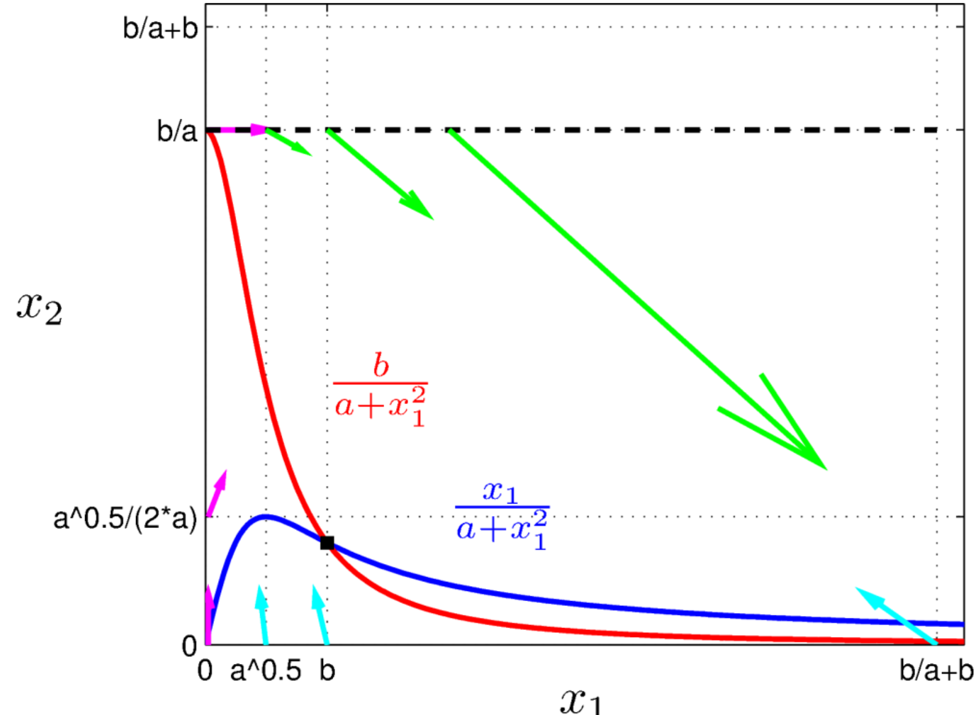
When $\boldsymbol{x_1 = b}$, we have:

$\boldsymbol{\dot{x}_1 = \frac{b^3}{a}, \dot{x}_2 = -\frac{b^3}{a}}$

(the slope of the vector field is -1, i.e. the angle is -45º at point $(\boldsymbol{b, b/a})$

In general, for large values of x_1 we have that : $\begin{aligned} \dot{x}_1 &\approx x_1^2 x_2 \\ \dot{x}_2 &\approx -x_1^2 x_2 \end{aligned}$





From the point (b,b/a) we move while following the vector field (slope =-1).

The equation of this line is:
$x_2 = -x_1 + n$, and to find the intercept, we note that the line passes through the point (b,b/a), i.e. $\frac{b}{a} = -b + n$, i.e. $n = \frac{b}{a} + b$

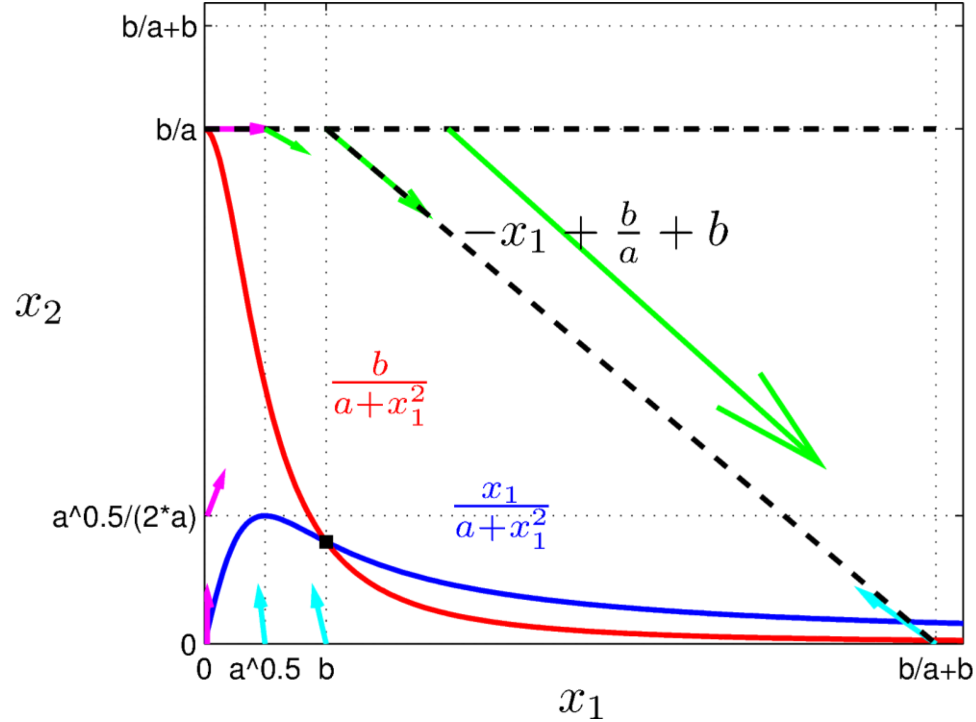
All we now have to do is to show that along this line, the vector field points toward the suspected region $\mathcal{D}$.

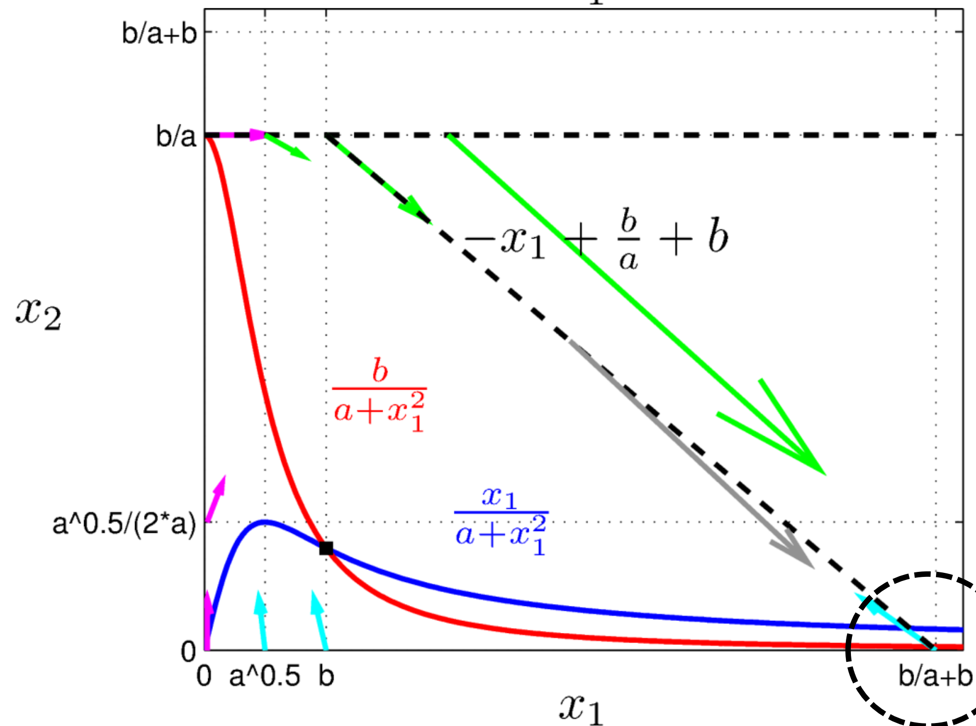
From the dynamics:

$$\dot{x}_1 = -x_1 + \underbrace{ax_2 + x_1^2 x_2}_{\alpha>0} > 0$$

$$\dot{x}_2 = b - ax_2 - x_1^2 x_2 = b - \alpha < 0$$

Therefore, if $x_1 > b$ (until dashed line intersects the blue nullcline), we have: $|\dot{x}_1| < |\dot{x}_2|$
(e.g. $|\dot{x}_1| = |b - \alpha| - \varepsilon$)

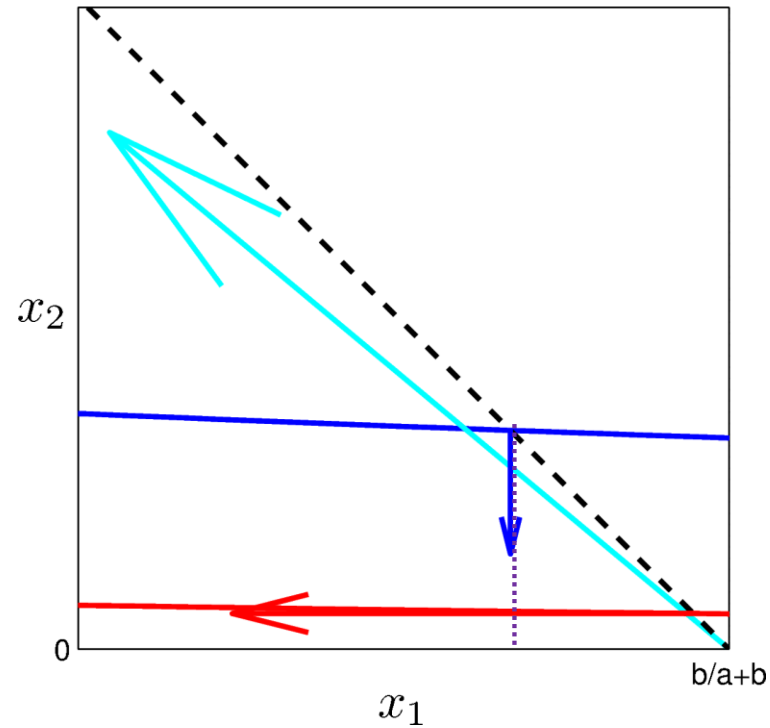


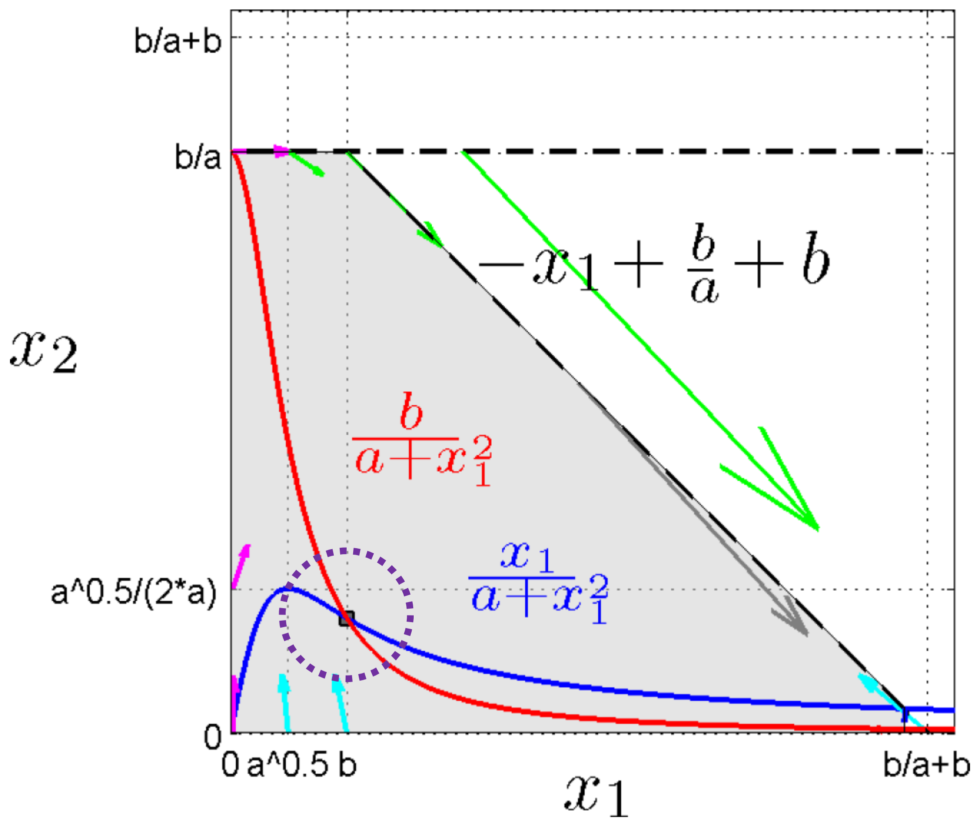


Zoom in on the circle and draw a vertical line straight from the intersection of the dashed line and blue nullcline.

Due to the smoothness of the vector field, the trajectories keep pointing inward.

Can we conclude that the region $\mathcal{D}$ (shaded) is a trapping region?





Not yet! We need to cut the equilibrium out and prove that it is some sort of repeller (unstable equilibrium).

Linearize the system:

$$\dot{x}_1 = -x_1 + ax_2 + x_1^2 x_2$$
$$\dot{x}_2 = b - ax_2 - x_1^2 x_2$$

The Jacobian matrix is:

$$Df(x) = \begin{bmatrix} -1 + 2x_1x_2 & a + x_1^2 \\ -2x_1x_2 & -(a + x_1^2) \end{bmatrix}$$

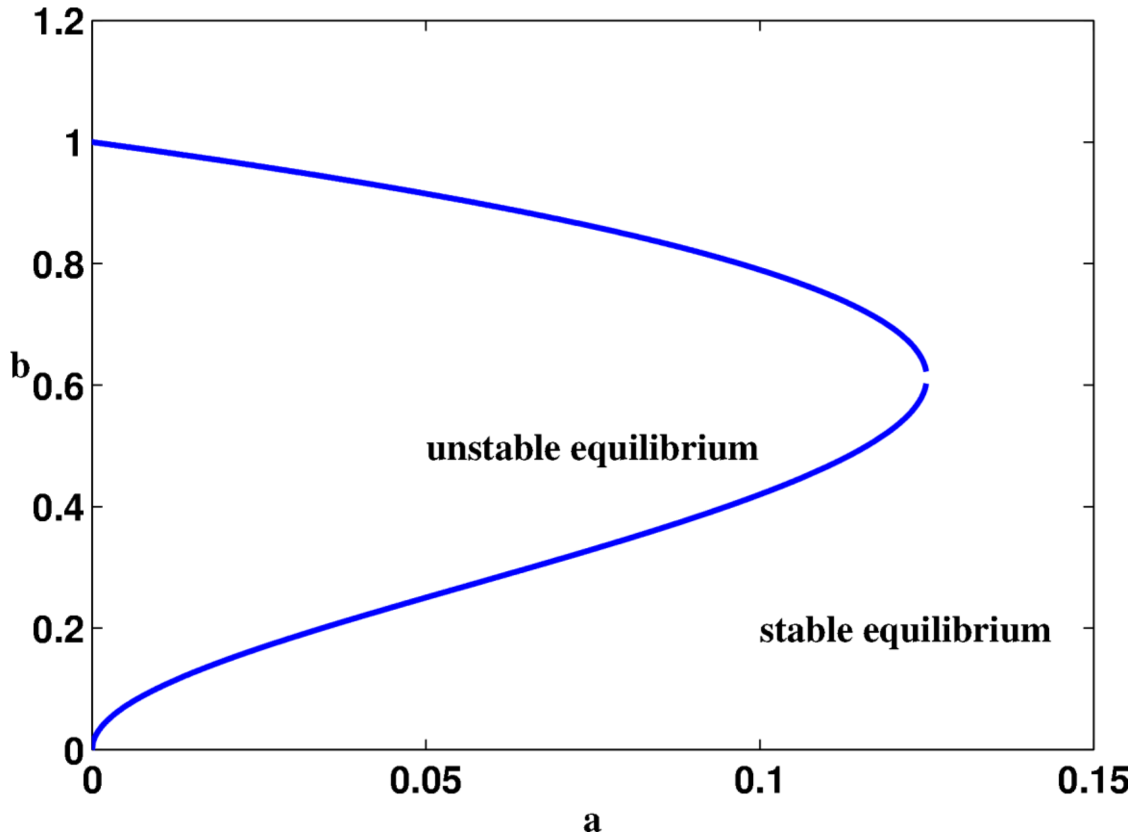
At equilibrium point:

$$Df\left(b, \frac{b}{a+b^2}\right) = A = \begin{bmatrix} -1 + 2\dfrac{b^2}{a+b^2} & a + b^2 \\ -2\dfrac{b^2}{a+b^2} & -(a+b^2) \end{bmatrix}$$

Note that:

$$\det(A) = \Delta = a + b^2 > 0$$
$$\mathrm{tr}(A) = \tau = -1 + 2\frac{b^2}{a+b^2} - (a + b^2)$$
$$\mathrm{tr}(A) = \tau = -\frac{b^4 + (2a-1)b^2 + (a^2 + a)}{a + b^2}$$

Hence, the equilibrium is unstable if $\tau > 0$ and stable if $\tau < 0$. The division line is $\tau = 0$, i.e. $b^2 = \frac{1-2a\pm\sqrt{1-8a}}{2}$


1.2
1
0.8
0.6
0.4
0.2
0
b
0
0.05
0.1
0.15
a
unstable equilibrium
stable equilibrium