Elementary Bifurcation Theory

Intuition: $\dot{x}(t) = f(x(t), \underbrace{\theta}_{\textbf{parameter}})$ -nonlinear time invariant system

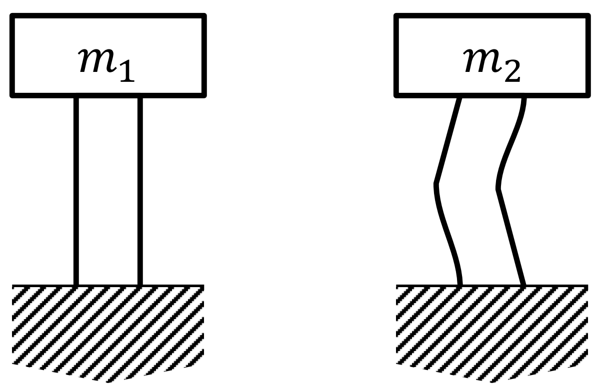
The qualitative structure of the vector field $f(x(t), \theta)$ may change as the parameter θ changes.

In particular:

- the equilibrium points may be created or destroyed
- the stability properties of equilibria may change

These qualitative changes in the dynamics of the system are called <u>bifurcations</u>, and the critical values of the parameter θ are called <u>bifurcation points</u>.

Example: buckling of a beam



$m_2 > m_1$

parameter: mass

Example: insect outbreak

$$\frac{dx}{dt} = \underbrace{rx\left(1-\frac{x}{k}\right)}_{\text{logistic growth}} - \underbrace{\frac{x^2}{1+x^2}}_{\text{death}}$$

r − growth rate
k − carrying capacity

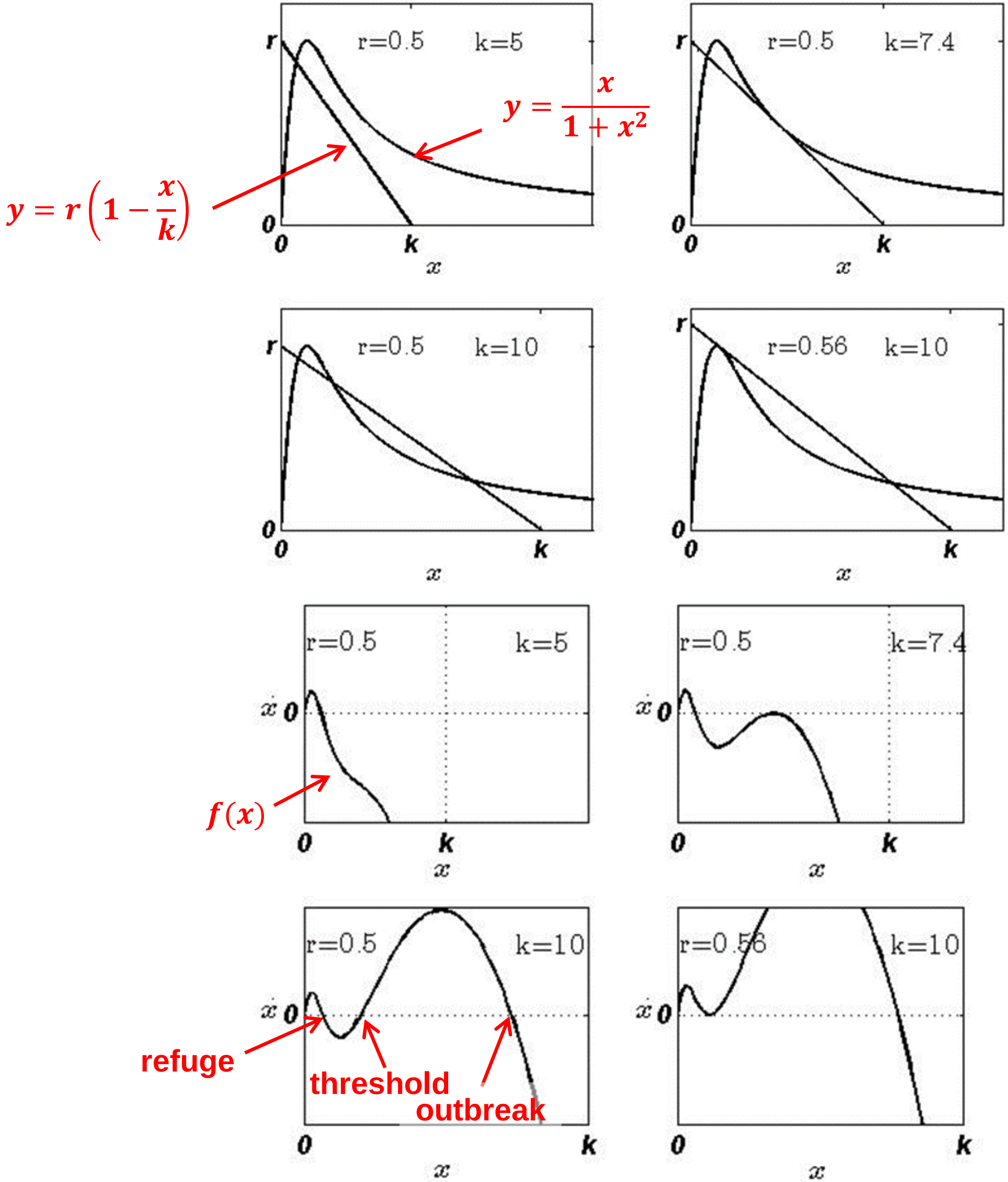
Equilibrium points:

$$\frac{dx^\star}{dt} = 0$$

$$rx\left(1-\frac{x}{k}\right) - \frac{x^2}{1+x^2} = 0$$

$$x\left[r\left(1-\frac{x}{k}\right) - \frac{x}{1+x^2}\right] = 0$$

$$\boxed{x^\star = 0} \text{ or } \boxed{r\left(1-\frac{x^\star}{k}\right) = \frac{x^\star}{1+{x^\star}^2}}$$


r=0.5
k=5
k=7.4
k=10
r=0.56
$y = \frac{x}{1+x^2}$
$y = r\left(1 - \frac{x}{k}\right)$
$f(x)$
refuge
threshold
outbreak

Bifurcation—qualitative changes in the behavior of the system depending on the choice of parameters.

Bifurcations can be:

1. Local (they can be analyzed using only local stability properties of equilibria, periodic orbits, or other invariant sets)
2. Global (they involve larger regions of the state space—not just the neighborhood of equilibria)

Local bifurcations can be:

- Static
- Dynamic

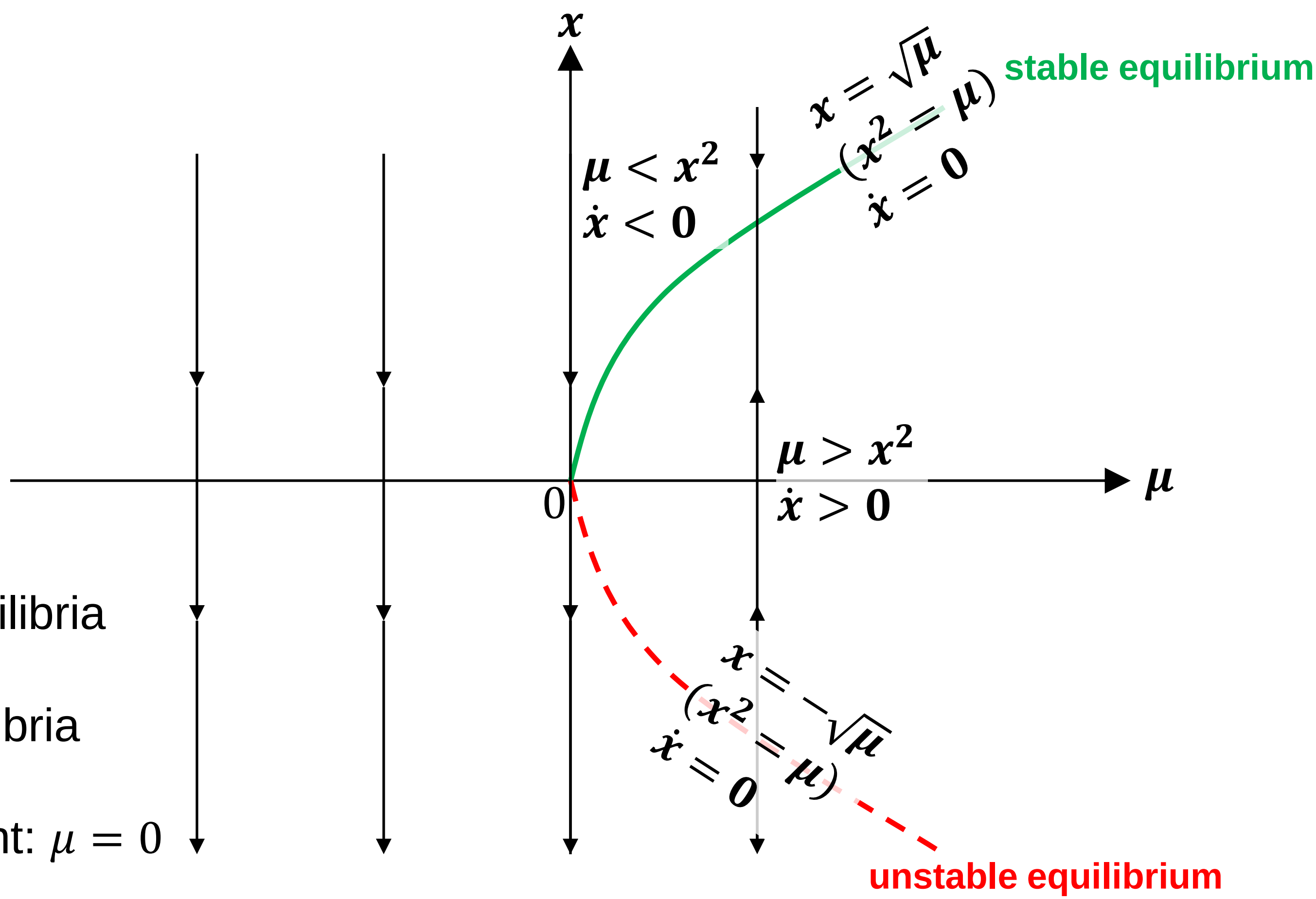
Static bifurcations—mechanisms by which equilibria are created or destroyed.

Dynamic bifurcations—mechanisms by which the stability of equilibria is changed.

Example: Saddle-node bifurcation (static)

$\dot{x} = \mu - x^2$

Equlibrium: $x^2 = \mu \Rightarrow \boxed{x^* = \pm\sqrt{\mu}\ (\mu > 0)}$



$\mu < 0$—no equilibria

$\mu > 0$—2 equilibria

bifurcation point: $\mu = 0$

Another way to see this is to linearize: $\dot{x} = \mu - x^2$

$Df(x) = -2x$

$Df(\sqrt{\mu}) = -2\sqrt{\mu} < 0$—stable. Note: $\mu > 0$

$Df(-\sqrt{\mu}) = 2\sqrt{\mu} > 0$—unstable. Note: $\mu > 0$

$\mu < 0$—no equilibria

Example: Transcritical bifurcation (dynamic)

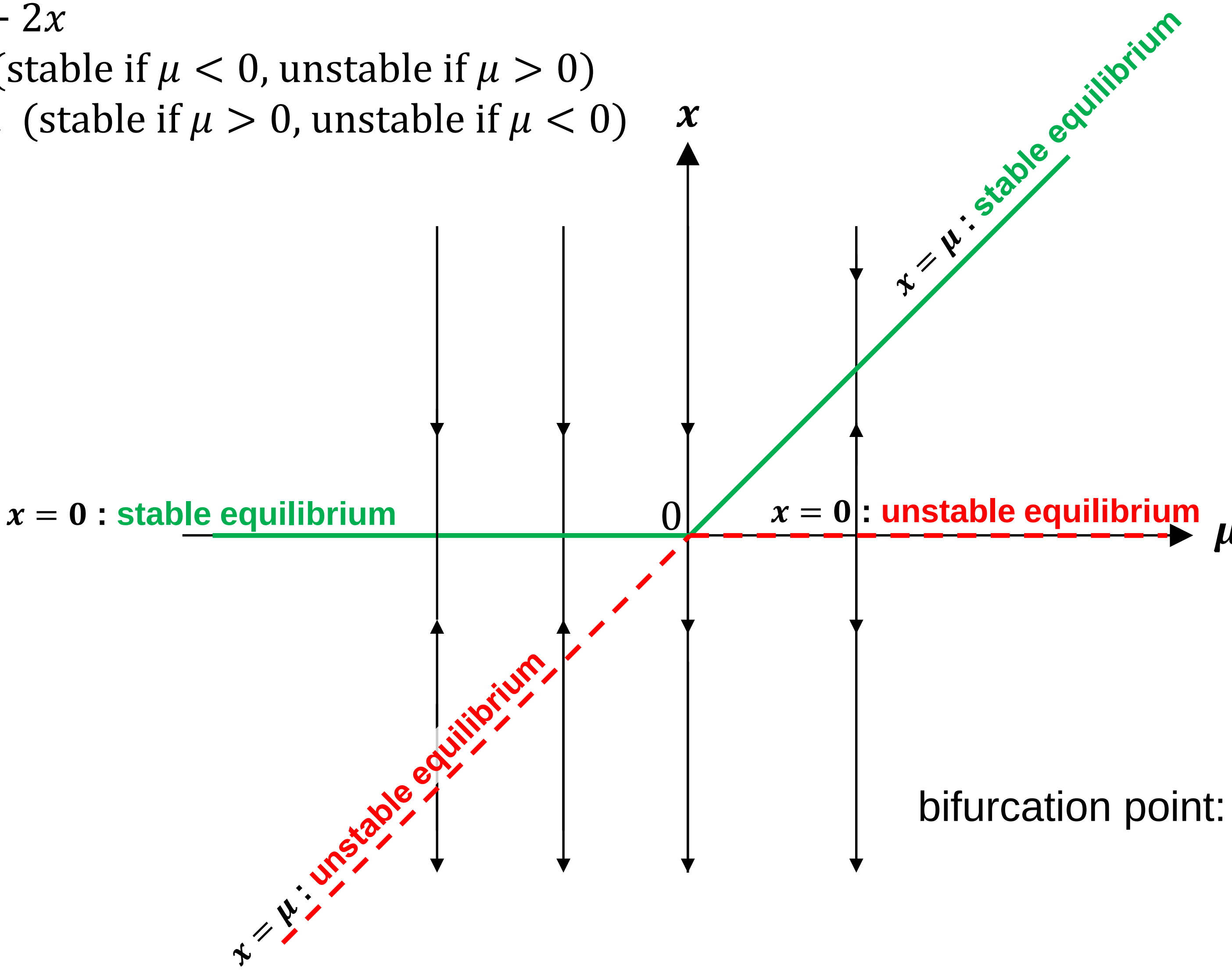
$\dot{x} = \mu x - x^2$

Equlibrium: $x^2 = \mu x \Rightarrow \boxed{x = 0}$ or $\boxed{x = \mu}$

$Df(x) = \mu - 2x$

$Df(0) = \mu$ (stable if $\mu < 0$, unstable if $\mu > 0$)

$Df(\mu) = -\mu$ (stable if $\mu > 0$, unstable if $\mu < 0$)



bifurcation point: $\mu = 0$

Example: Pitchfork bifurcation (static & dynamic)

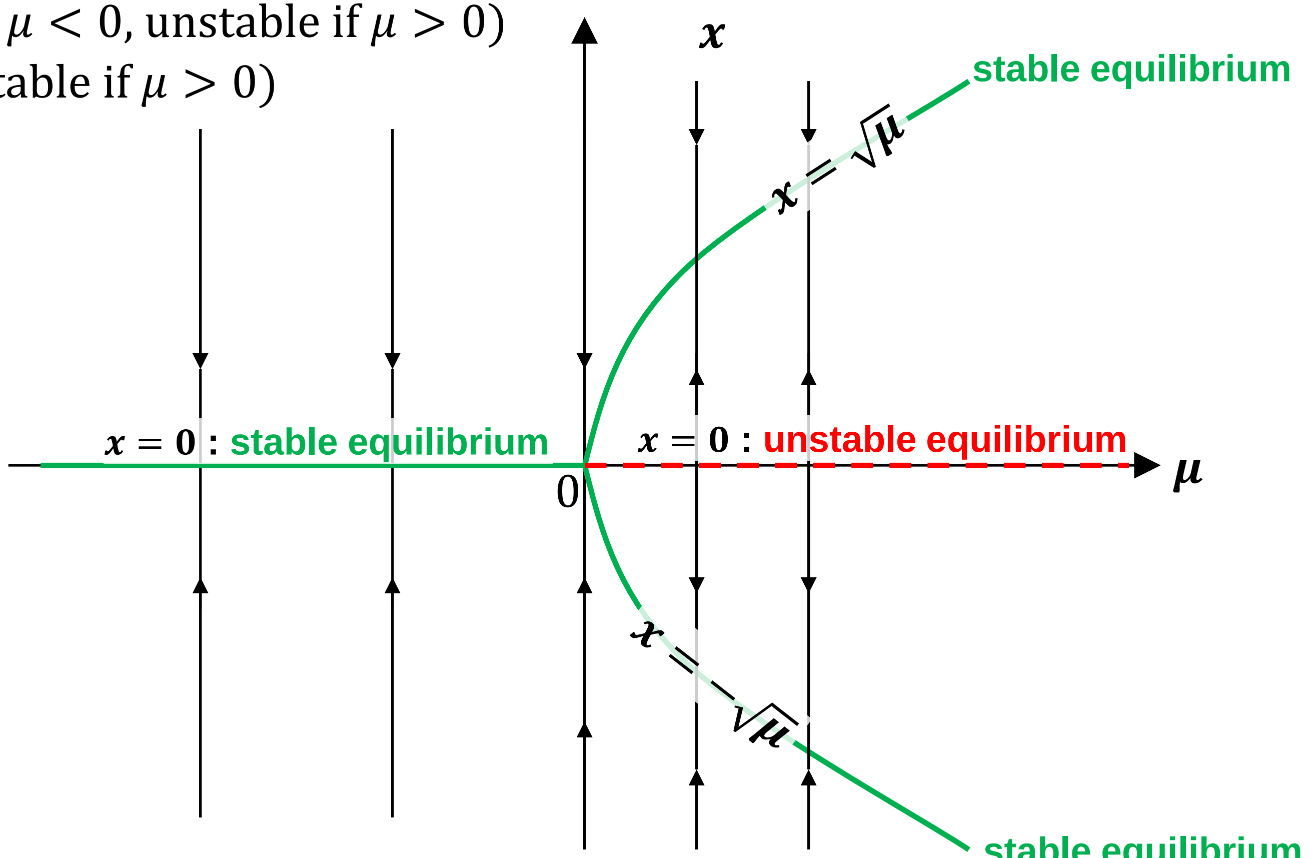
$\dot{x} = \mu x - x^3$

Equlibrium: $x^3 = \mu x \Rightarrow \boxed{x = 0}$ or $\boxed{x^2 = \mu}$

$Df(x) = \mu - 3x^2$

$Df(0) = \mu$ (stable if $\mu < 0$, unstable if $\mu > 0$)

$Df(\pm\sqrt{\mu}) = -2\mu$ (stable if $\mu > 0$)



bifurcation point
@ $\mu = 0$

Example: Hopf bifurcation (dynamic)

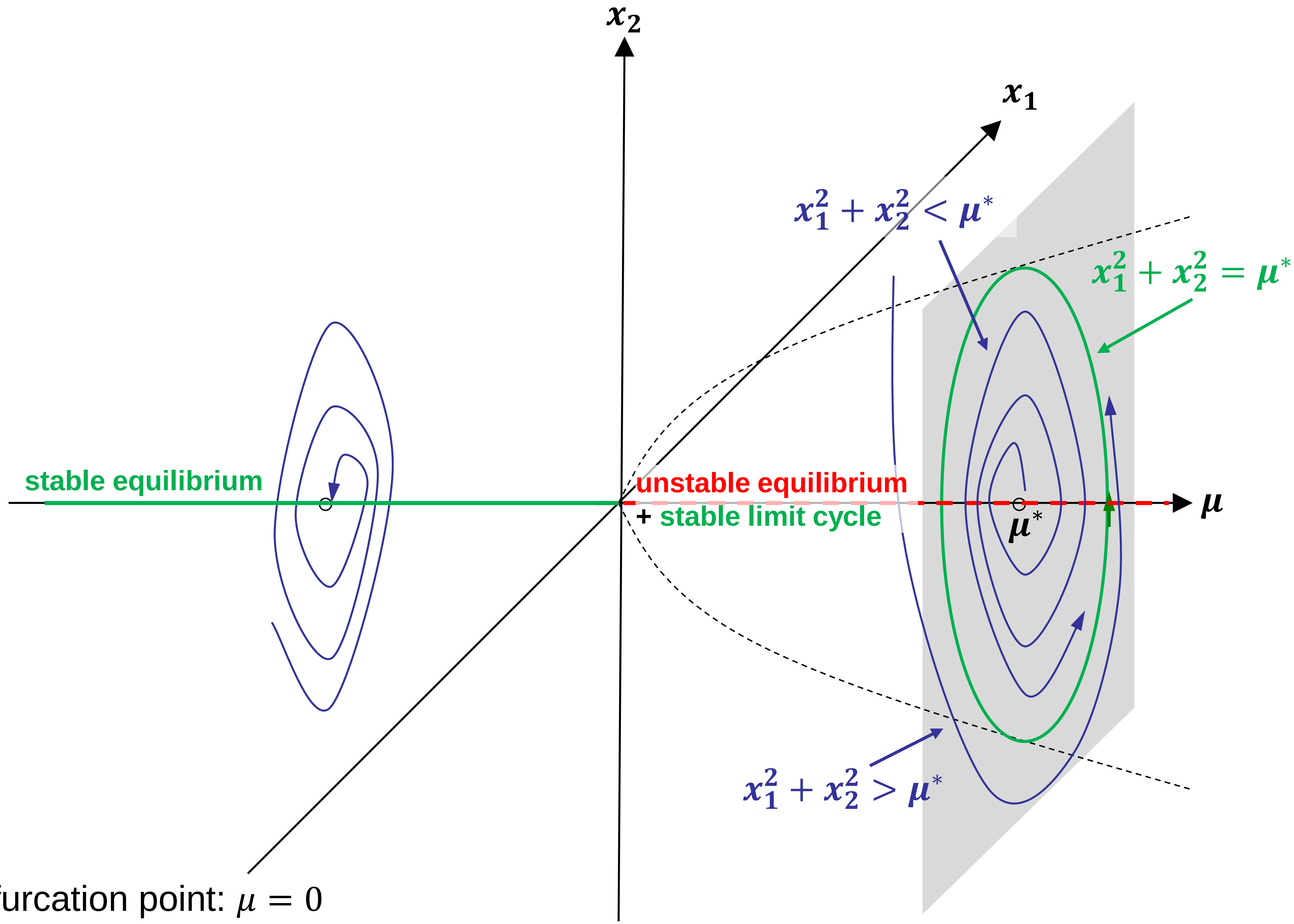
$$\dot{x}_1 = -x_2 + x_1\left(\mu - (x_1^2 + x_2^2)\right)$$
$$\dot{x}_2 = x_1 + x_2\left(\mu - (x_1^2 + x_2^2)\right)$$

Polar coordinates: $x_1 = \rho\cos(\varphi)$; $x_2 = \rho\sin(\varphi)$
Straightforward calculations:

$$\dot{\rho} = \rho(\mu - \rho^2)$$
$$\dot{\varphi} = 1$$

Equilibrium: $\dot{\rho} = 0 \Rightarrow \boxed{\rho = 0}$ or $\boxed{\rho^2 = \mu}$ (note: $\mu > 0$)
$Df(\rho) = \mu - 3\rho^2$
$Df(0) = \mu$ –stable if $\mu < 0$ and unstable if $\mu > 0$.

If $\mu < 0$ there is only a single equilibrium point $\rho = 0$ $(x_1 = 0; x_2 = 0)$
If $\mu > 0$ there is a single equilibrium point $\rho = 0$, *and a stable limit cycle at* $\rho = \sqrt{\mu}$.



bifurcation point: $\mu = 0$