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| **44715: Optimization II** | **2019S** |

# Homework 1: Duality

*Name:* *ID:*

**Note**: *In all notes, bold face letters denote vectors.*

1. The weak duality lemma is proved for symmetric form in the class (and note). Prove the weak duality lemma for the asymmetric form. More precisely, suppose the asymmetric form of primal problem and dual problems in the following table 17.2. Prove that for any feasible solutions $\mathbf{x}$ and $\boldsymbol{\lambda}$ for the primal

**Table 17.1** Symmetric Form of Duality

| Primal | | Dual | |
|---|---|---|---|
| minimize | $\boldsymbol{c}^\top \boldsymbol{x}$ | maximize | $\boldsymbol{\lambda}^\top \boldsymbol{b}$ |
| subject to | $\boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}$ | subject to | $\boldsymbol{\lambda}^\top \boldsymbol{A} \leq \boldsymbol{c}^\top$ |
| | $\boldsymbol{x} \geq \mathbf{0}$ | | $\boldsymbol{\lambda} \geq \mathbf{0}$ |

**Table 17.2** Asymmetric Form of Duality

| Primal | | Dual | |
|---|---|---|---|
| minimize | $\boldsymbol{c}^\top \boldsymbol{x}$ | maximize | $\boldsymbol{\lambda}^\top \boldsymbol{b}$ |
| subject to | $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ | subject to | $\boldsymbol{\lambda}^\top \boldsymbol{A} \leq \boldsymbol{c}^\top$ |
| | $\boldsymbol{x} \geq \mathbf{0}$ | | |

and dual problems respectively, they satisfy

$$\mathbf{c}^T \mathbf{x} \geq \boldsymbol{\lambda}^T \mathbf{b} \tag{1.1}$$

2. Consider the following LP,

$$\begin{aligned} \text{minimize} \quad & 4x_1 + 3x_2 \\ \text{subject to} \quad & 5x_1 + x_2 \geq 11 \\ & 2x_1 + x_2 \geq 8 \\ & x_1 + 2x_2 \geq 7 \\ & x_1, x_2 \geq 0 \end{aligned} \tag{1.2}$$

Write down the dual problem (no need to solve it).

3. Transform the above problem into the standard form, find the dual problem.

4. Consider the LP

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}^T \mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \end{aligned} \tag{1.3}$$

Find the dual problem first. Suppose $\mathbf{b} = \mathbf{0}$ and there exists a vector $\mathbf{z} \geq \mathbf{0}$ such that $A^T \mathbf{z} + \mathbf{c} = \mathbf{0}$. Does above problem have an optimal feasible solution? Why?

5. Consider the LP

$$\begin{array}{rl} \text{minimize} & x_1 + x_2 + \cdots + x_n \\ \text{subject to} & a_1 x_1 + \cdots a_n x_n = 1 \\ & x_1, \ldots, x_n \geq 0, \end{array} \tag{1.4}$$

   where $0 < a_1 < a_2 < \cdots < a_n$.

   (a) Write down the dual problem and find the solution to the dual in terms of $a_1, \cdots, a_n$.

   (b) Use the duality theorem to find the optimal objective function's value to the primal.

   (c) Apply the Complementary Slackness Condition to find an optimal solution to the primal.

6. Consider the problem

$$\begin{array}{rl} \text{minimize} & \mathbf{c}^T \mathbf{x} \\ \text{subject to} & \mathbf{x} \geq 0 \end{array} \tag{1.5}$$

   Use the duality theorem to show that: the above problem has a feasible solution if and only if $\mathbf{c} \geq 0$. If a feasible solution exists, then $\mathbf{x} = \mathbf{0}$ is an optimal solution.