Homework to Chapters 30,31

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① Let V, W be two finite dimensional vector spaces over a field F and let $\varphi: V \to W$ a surjective linear map. Let $\{v_1, \dots, v_k\}$ be a basis in $Ker(\varphi) \subset V$.

Show that it can be extended to some basis $\{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$ of V and that $\{\varphi(v_{k+1}), \dots, \varphi(v_n)\}$ will be a basis of W

② Let $\varphi: V \to W$ be an injective linear map of finite dimensional vector spaces. Show that there exists a linear map $\psi: W \to V$ such that the composition $V \xrightarrow{\varphi} W \xrightarrow{\psi} V$ is the identity map on V.

③ Let $S = \{v_1, \dots, v_k\} \subset V$ be a finite subset of elements in an <u>abelian group</u> V. We will say that S <u>spans V over $\mathbb{Z}$</u> if any $v \in V$ is a linear combination

$$v = a_1 v_1 + \dots + a_k v_k \quad \text{with} \quad a_i \in \mathbb{Z}.$$

Give an example of an abelian group V and a set of two vectors $S = \{v_1, v_2\} \subset V$ such that

(i) S <u>spans V over $\mathbb{Z}$</u>, and

(ii) $a_1 v_1 + a_2 v_2 = 0$ for some $a_1 \neq 0, a_2 \neq 0$ in $\mathbb{Z}$, and

(iii) neither $\{v_1\}$ nor $\{v_2\}$ spans V over $\mathbb{Z}$.

(<u>Remark</u> This shows that the pruning lemma does not work over $\mathbb{Z}$. The reason, of course, is that $\mathbb{Z}$ is not a field)

④ Let $F \subset E$ be a field extension of degree $[E:F] = 5$. Let $\alpha \in E$ be an element such that $\alpha^2 \in F$. Show that also $\alpha \in F$.

⑤ Prove that any $b \in \mathbb{Q}(\sqrt[3]{7})$ is algebraic over $\mathbb{Q}$.

(5) Prove that any $\beta \in \mathbb{Q}(\sqrt[3]{7})$ is algebraic over $\mathbb{Q}$.

(6) Find the algebraic closure of $\mathbb{C}$ in $\mathbb{C}(x)$.