Math 120B: problems on and around UFDs

1. Find the content of $f(x) = 12x^2 - 72x + 18$ in $\mathbb{Z}[x]$ and its decomposition into product of irreducibles. Prove that your factors are indeed irreducible.

2. Let D be a unique factorization domain in which all irreducible elements are associates. If p is any such element, show that any ideal in D is of the form (p^n) for some $n \geq 0$.

3. Suppose that R is a unique factorization domain which is NOT a principal ideal domain.

   (i) Show that R must have at least two (nonassociate) irreducible elements (use the previous problem).

   (ii) Show that R must have a nonprincipal maximal ideal.

4. Factor $x^4 + 4y^4$ into a product of irreducibles in $\mathbb{Q}[x, y]$ and prove that your factors are indeed irreducible.

5. Let D be an integral domain and $D[x]$ the polynomial ring over D. Show that if every nonzero prime ideal of $D[x]$ is a maximal ideal, then D is a field.

6. Let a, b be two nonzero elements in a unique factorization domain D. Similarly to a greatest common divisor, define the least common multple of a, b and show that it exists. Explain how it can be computed, given decompositon of a and b into products of irreducibles.