Math 120B: Sample Final

Closed book, closed notes, no calculators. Each problem is worth 10 points. Time: 80 minutes. Please explain your solutions. Just giving an answer is not enough.

1. Suppose that $I \subset \mathbb{Z}[x]$ is an ideal and there is a prime $p \in \mathbb{Z}$ which is in I. Show that I can be generated by two elements, i.e. there exists $z \in I$ such that $I = \{r_1 p + r_2 z | r_1, r_2 \in \mathbb{Z}[x]]\}$.

2. Is it true that the intersection of two prime ideals is always a prime ideal? Explain.

3. Let F be a field and assume that $R = F[x]/(f(x))$ is an integral domain for some polynomial $f(x)$. Show that in fact R is a field.

4. Let $F \subset E$ be a field extension of finite degree and assume that the degree $[E : F] = p$ is a prime. Show that for any $\alpha \in E$ either $F(\alpha) = F$ or $F(\alpha) = E$.

5. Let $f_1, f_2 \in F[x]$ be two polynomials (and F is a field). Let $g = gcd(f_1, f_2)$. Show that the ideal I generated by f_1, f_2 (i.e.

$$I = \{h_1 f_1 + h_2 f_2 \quad | \quad h_1, h_2 \in F[x]\}$$

satisfies $I = (g(x))$.

6. Construct a field with 32 elements. Prove that what you have constructed indeed has 32 elements and that it is indeed a field.