Prediction and Search in Probabilistic Worlds

# Markov Systems, Markov Decision Processes, and Dynamic Programming

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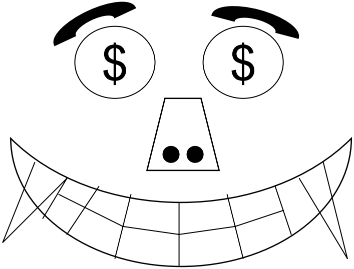
# Discounted Rewards

An assistant professor gets paid, say, 20K per year.

How much, in total, will the A.P. earn in their life?

20 + 20 + 20 + 20 + 20 + … = Infinity




What's wrong with this argument?

# Discounted Rewards

"A reward (payment) in the future is not worth quite as much as a reward now."

- Because of chance of obliteration
- Because of inflation

Example:

Being promised $10,000 next year is worth only 90% as much as receiving $10,000 right now.

Assuming payment *n* years in future is worth only $(0.9)^n$ of payment now, what is the AP's Future Discounted Sum of Rewards ?

# Discount Factors

People in economics and probabilistic decision-making do this all the time.

The "Discounted sum of future rewards" using discount factor $\gamma$" is

(reward now) +

$\gamma$ (reward in 1 time step) +

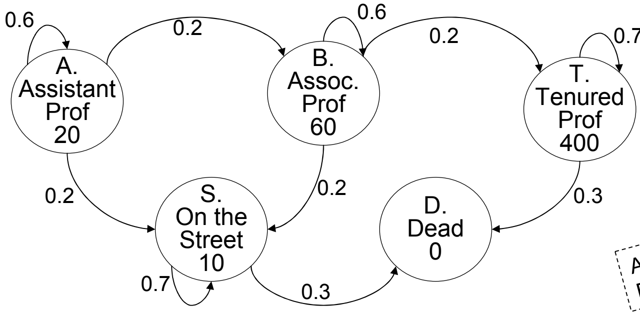
$\gamma^2$ (reward in 2 time steps) +

$\gamma^3$ (reward in 3 time steps) +

:

: (infinite sum)

# The Academic Life




Assume Discount Factor $\gamma = 0.9$

Define:

$J_A$ = Expected discounted future rewards starting in state A

$J_B$ = Expected discounted future rewards starting in state B

$J_T$ = " " " " " " " T

$J_S$ = " " " " " " " S

$J_D$ = " " " " " " " D

How do we compute $J_A$, $J_B$, $J_T$, $J_S$, $J_D$ ?

# Computing the Future Rewards of an Academic

# A Markov System with Rewards…

- Has a set of states $\{S_1\ S_2 \cdots S_N\}$
- Has a transition probability matrix

$$P = \begin{pmatrix} P_{11}\ P_{12} \cdots P_{1N} \\ P_{21} \\ \vdots \\ P_{N1} \quad \cdots \quad P_{NN} \end{pmatrix} \qquad P_{ij} = \text{Prob}(\text{Next} = S_j \mid \text{This} = S_i)$$

- Each state has a reward. $\{r_1\ r_2 \cdots r_N\}$
- There's a discount factor $\gamma$ . $0 < \gamma < 1$

On Each Time Step …

0. Assume your state is $S_i$
1. You get given reward $r_i$
2. You randomly move to another state
   $$P(\text{NextState} = S_j \mid \text{This} = S_i) = P_{ij}$$
3. All future rewards are discounted by $\gamma$

# Solving a Markov System

Write $J^*(S_i)$ = expected discounted sum of future rewards starting in state $S_i$

$J^*(S_i) = r_i + \gamma$ x (Expected future rewards starting from your next state)

$$= r_i + \gamma(P_{i1}J^*(S_1)+P_{i2}J^*(S_2)+ \cdots P_{iN}J^*(S_N))$$

Using vector notation write

$$\underline{J}=\begin{pmatrix} J^*(S_1) \\ J^*(S_2) \\ : \\ J^*(S_N) \end{pmatrix} \qquad \underline{R}=\begin{pmatrix} r_1 \\ r_2 \\ : \\ r_N \end{pmatrix} \qquad \underline{P}=\begin{pmatrix} P_{11} & P_{12} & \cdot\cdot & P_{1N} \\ P_{21} & \ddots & & \\ : & & & \\ P_{N1} & P_{N2} & \cdot\cdot & P_{NN} \end{pmatrix}$$

Question: can you invent a closed form expression for $\underline{J}$ in terms of $\underline{R}$ $\underline{P}$ and $\gamma$ ?

# Solving a Markov System with Matrix Inversion

- Upside: You get an exact answer
- Downside:

# Solving a Markov System with Matrix Inversion

- Upside: You get an exact answer

- Downside: If you have 100,000 states you're solving a 100,000 by 100,000 system of equations.

# Value Iteration: another way to solve a Markov System

Define

$J^1(S_i)$ = Expected discounted sum of rewards over the next 1 time step.

$J^2(S_i)$ = Expected discounted sum rewards during next 2 steps

$J^3(S_i)$ = Expected discounted sum rewards during next 3 steps

:

$J^k(S_i)$ = Expected discounted sum rewards during next *k* steps

$J^1(S_i) =$ (what?)

$J^2(S_i) =$ (what?)

:

$J^{k+1}(S_i) =$ (what?)

# Value Iteration: another way to solve a Markov System

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:

$J^k(S_i)$ = Expected discounted sum rewards during next *k* steps

N = Number of states

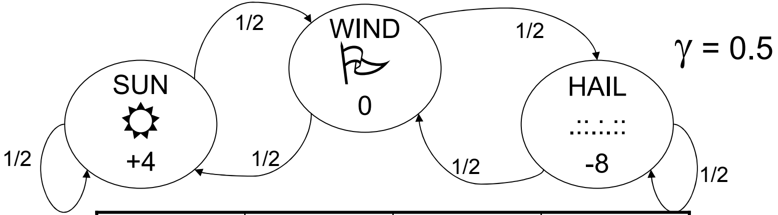
$J^1(S_i) = r_i$ (what?)

$J^2(S_i) = r_i + \gamma \sum_{j=1}^{N} p_{ij} J^1(s_j)$ (what?)

:

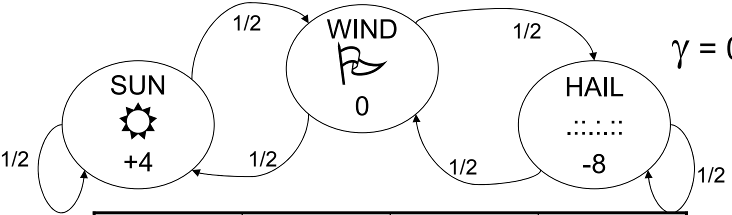
$J^{k+1}(S_i) = r_i + \gamma \sum_{j=1}^{N} p_{ij} J^k(s_j)$ (what?)

# Let's do Value Iteration




| k | $J^k$(SUN) | $J^k$(WIND) | $J^k$(HAIL) |
|---|---|---|---|
| 1 | | | |
| 2 | | | |
| 3 | | | |
| 4 | | | |
| 5 | | | |

# Let's do Value Iteration




$\gamma = 0.5$

| k | $J^k$(SUN) | $J^k$(WIND) | $J^k$(HAIL) |
|---|---|---|---|
| 1 | 4 | 0 | -8 |
| 2 | 5 | -1 | -10 |
| 3 | 5 | -1.25 | -10.75 |
| 4 | 4.94 | -1.44 | -11 |
| 5 | 4.88 | -1.52 | -11.11 |

# Value Iteration for solving Markov Systems

- Compute $J^1(S_i)$ for each $j$
- Compute $J^2(S_i)$ for each $j$

:

- Compute $J^k(S_i)$ for each $j$

As $k \to \infty$ $J^k(S_i) \to J^*(S_i)$ . Why?

When to stop? When

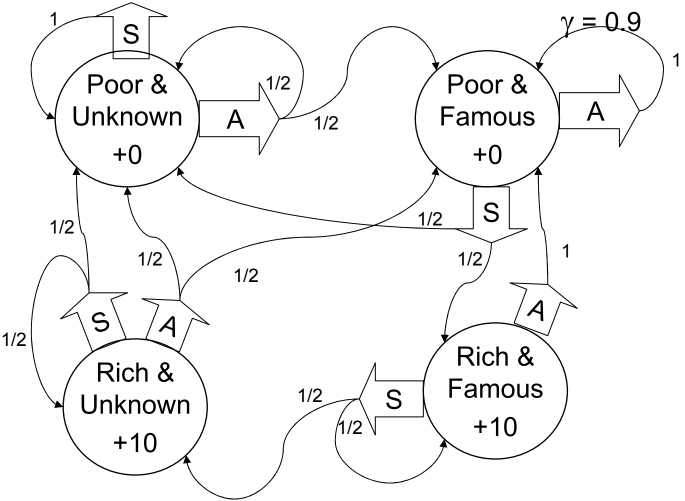
$$\max_i \left| J^{k+1}(S_i) - J^k(S_i) \right| < \xi$$

This is faster than matrix inversion ($N^3$ style)
if the transition matrix is sparse

# A Markov Decision Process

You run a startup company.

In every state you must choose between Saving money or Advertising.



# Markov Decision Processes

An MDP has…

- A set of states $\{s_1 \cdots S_N\}$
- A set of actions $\{a_1 \cdots a_M\}$
- A set of rewards $\{r_1 \cdots r_N\}$ (one for each state)
- A transition probability function

$$P_{ij}^{k} = \text{Prob}\left(\text{Next} = j \middle| \text{This} = i \text{ and I use action } k\right)$$

On each step:

0. Call current state $S_i$
1. Receive reward $r_i$
2. Choose action $\in \{a_1 \cdots a_M\}$
3. If you choose action $a_k$ you'll move to state $S_j$ with probability $P_{ij}^{k}$
4. All future rewards are discounted by $\gamma$

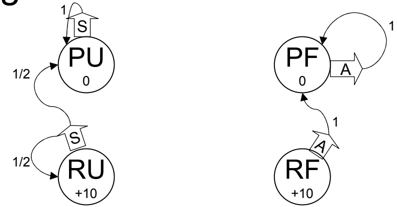
# A Policy

A policy is a mapping from states to actions.

Examples

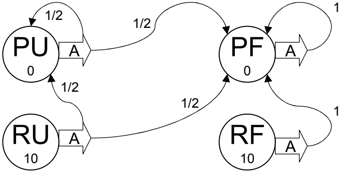
Policy Number 1:

| STATE → | ACTION |
|---|---|
| PU | S |
| PF | A |
| RU | S |
| RF | A |




Policy Number 2:

| STATE → | ACTION |
|---|---|
| PU | A |
| PF | A |
| RU | A |
| RF | A |




- How many possible policies in our example?
- Which of the above two policies is best?
- How do you compute the optimal policy?

# Interesting Fact

For every M.D.P. there exists an optimal policy.

It's a policy such that for every possible start state there is no better option than to follow the policy.

(Not proved in this lecture)

# Computing the Optimal Policy

Idea One:

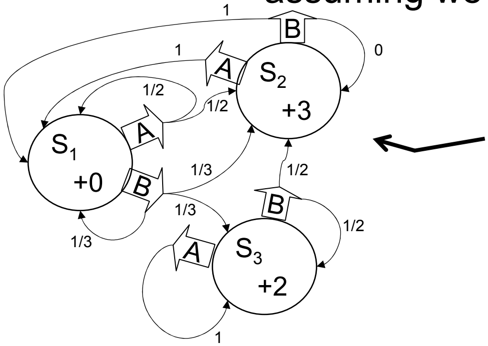
Run through all possible policies.

Select the best.

What's the problem ??

# Optimal Value Function

Define $J^*(S_i)$ = Expected Discounted Future Rewards, starting from state $S_i$, assuming we use the optimal policy




Question

What (by inspection) is an optimal policy for that MDP?

(assume $\gamma = 0.9$)

What is $J^*(S_1)$ ?
What is $J^*(S_2)$ ?
What is $J^*(S_3)$ ?

# Computing the Optimal Value Function with Value Iteration

Define

$J^k(S_i)$ = Maximum possible expected sum of discounted rewards I can get if I start at state $S_i$ and I live for *k* time steps.

Note that $J^1(S_i) = r_i$

# Let's compute $J^k(S_i)$ for our example

| k | $J^k(PU)$ | $J^k(PF)$ | $J^k(RU)$ | $J^k(RF)$ |
|---|---|---|---|---|
| 1 | | | | |
| 2 | | | | |
| 3 | | | | |
| 4 | | | | |
| 5 | | | | |
| 6 | | | | |

# Let's compute $J^k(S_i)$ for our example

| k | $J^k(PU)$ | $J^k(PF)$ | $J^k(RU)$ | $J^k(RF)$ |
|---|---|---|---|---|
| 1 | 0 | 0 | 10 | 10 |
| 2 | 0 | 4.5 | 14.5 | 19 |
| 3 | 2.03 | 6.53 | 25.08 | 18.55 |
| 4 | 3.852 | 12.20 | 29.63 | 19.26 |
| 5 | 7.22 | 15.07 | 32.00 | 20.40 |
| 6 | 10.03 | 17.65 | 33.58 | 22.43 |

# Bellman's Equation

$$J^{n+1}(S_i) = \max_k \left[ r_i + \gamma \sum_{j=1}^{N} P_{ij}^k J^n(S_j) \right]$$

## Value Iteration for solving MDPs

- Compute $J^1(S_i)$ for all *i*
- Compute $J^2(S_i)$ for all *i*
- :
- Compute $J^n(S_i)$ for all *i*

…..until converged

$$\left[ \text{converged when } \max_i \left| J^{n+1}(S_i) - J^n(S_i) \right| \langle \xi \right]$$

…Also known as

Dynamic Programming

# Finding the Optimal Policy

1. Compute $J^*(S_i)$ for all i using Value Iteration (a.k.a. Dynamic Programming)

2. Define the best action in state $S_i$ as

$$\arg\max_k \left[ r_i + \gamma \sum_j P_{ij}^k J^*\left(S_j\right) \right]$$

(Why?)

# Applications of MDPs

This extends the search algorithms of your first lectures to the case of probabilistic next states.

Many important problems are MDPs….

… Robot path planning

… Travel route planning

… Elevator scheduling

… Bank customer retention

… Autonomous aircraft navigation

… Manufacturing processes

… Network switching & routing

# Asynchronous D.P.

Value Iteration:

“Backup $S_1$”, “Backup $S_2$”, ···· “Backup $S_N$”,
then “Backup $S_1$”, “Backup $S_2$”, ····
repeat :
: There's no reason that you need to do the backups in order!

Random Order …still works. Easy to parallelize (Dyna, Sutton 91)

On-Policy Order

Simulate the states that the system actually visits.

Efficient Order

e.g. Prioritized Sweeping [Moore 93]
Q-Dyna [Peng & Williams 93]

# Policy Iteration

Another way to compute optimal policies

Write $\pi(S_i)$ = action selected in the $i$'th state. Then $\pi$ is a policy.

Write $\pi^t$ = $t$'th policy on $t$'th iteration

Algorithm:

$\pi^\circ$ = Any randomly chosen policy

$\forall i$ compute $J^\circ(S_i)$ = Long term reward starting at $S_i$ using $\pi^\circ$

$\pi_1(S_i)$ = $\arg\max_a \left[ r_i + \gamma \sum_j P_{ij}^a J^\circ(S_j) \right]$

$J_1$ = ….

$\pi_2(S_i)$ = ….

… Keep computing $\pi^1$ , $\pi^2$ , $\pi^3$ …. until $\pi^k = \pi^{k+1}$ . You now have an optimal policy.

# Policy Iteration & Value Iteration: Which is best ???

It depends.

Lots of actions? Choose Policy Iteration

Already got a fair policy? Policy Iteration

Few actions, acyclic? Value Iteration

Best of Both Worlds:

Modified Policy Iteration [Puterman]

…a simple mix of value iteration and policy iteration

3rd Approach

Linear Programming

# Time to Moan

What’s the biggest problem(s) with what we’ve seen so far?

# Dealing with large numbers of states

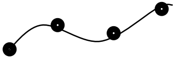
Don't use a Table…

| STATE | VALUE |
|---|---|
| $s_1$ | |
| $S_2$ | |
| : | |
| $S_{15122189}$ | |

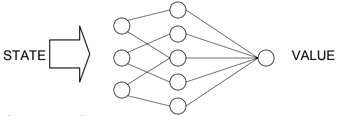
use…

(Generalizers)

Splines

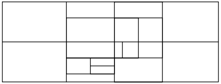


A Function Approximator



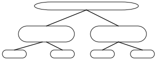

(Hierarchies)

Variable Resolution

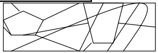


[Munos 1999]

Multi Resolution



Memory Based



# Function approximation for value functions

Polynomials → [Samuel, Boyan, Much O.R. Literature]

Neural Nets → [Barto & Sutton, Tesauro, Crites, Singh, Tsitsiklis]

Backgammon, Pole Balancing, Elevators, Tetris, Cell phones

Checkers, Channel Routing, Radio Therapy

Splines → Economists, Controls

**Downside:** All convergence guarantees disappear.

# Memory-based Value Functions

J("state") = J(most similar state in memory to "state")

or

Average J(20 most similar states)

or

Weighted Average J(20 most similar states)

[Jeff Peng, Atkenson & Schaal,

Geoff Gordon, ← proved stuff

Scheider, Boyan & Moore 98]

"Planet Mars Scheduler"

# Hierarchical Methods

Continuous State Space: "Split a state when statistically significant that a split would improve performance"

Discrete Space:

Chapman & Kaelbling 92, McCallum 95 (includes hidden state)

A kind of Decision Tree Value Function

Continuous Space

e.g. Simmons et al 83, Chapman & Knelbling 92, Mark Ring 94 …, Munos 96

with interpolation!

"Prove needs a higher resolution"

Moore 93, Moore & Atkeson 95

Multiresolution

A hierarchy with high level "managers" abstracting low level "servants"

<u>Many</u> O.R. Papers, Dayan & Sejnowski's Feudal learning, Dietterich 1998 (MAX-Q hierarchy) Moore, Baird & Kaelbling 2000 (airports Hierarchy)

# What You Should Know

- Definition of a Markov System with Discounted rewards
- How to solve it with Matrix Inversion
- How (and why) to solve it with Value Iteration
- Definition of an MDP, and value iteration to solve an MDP
- Policy iteration
- Great respect for the way this formalism generalizes the deterministic searching of the start of the class
- But awareness of what has been sacrificed.