Math 220A Fall 00 D.A. Martin

Mathematical Logic and Set Theory

1 Basic set theory

Iterative concept of set.

(a) Sets are formed in stages $0, 1, \ldots, s, \ldots$.

(b) For each stage s, there is a next stage $s + 1$.

(c) There is an "absolute infinity" of stages.

(d) V_s = the collection of all sets formed before stage s.

(e) $V_0 = \emptyset$ = the empty collection.

(f) V_{s+1} = the collection of (a) all sets belonging to V_s and (b) all subcollections of V_s not previously formed into sets.

Remarks. (1) A set is formed after its members. (2) V_s itself is formed as a set at stage s.

Formal language for talking about sets.

Symbols:

$v_0, v_1, v_2, \ldots$	variables
$=$	meaning "is identical with"
$\in$	meaning "is a member of"
$\neg$	meaning "not"
$\wedge$	meaning "and"
$\exists$	meaning "there is a"
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Formulas (inductive definition):

(i) If x and y are variables, then $x = y$ and $x \in y$ are (atomic) formulas.

(ii) If x is a variable and φ and ψ are formulas, then $\neg\varphi$, $(\varphi \wedge \psi)$, and $(\exists x)\,\varphi$ are formulas .

(iii) Nothing is a formula unless (i) and (ii) require it to be.

Free occurences of a variable in a formula:

(i) All occurrences of variables in atomic formulas $x \in y$ and $x = y$ are free.

(ii) An occurrence of x in $\neg\varphi$ is free in just in case the corresponding occurrence of x in φ is free.

(iii) An occurrence of x in $(\varphi \wedge \psi)$ is free in just in case the corresponding occurrence of x in φ or in ψ is free.

(iv) An occurrence of x in $(\exists y)\,\varphi$ is free in just in case x is not y and the corresponding occurrence of x in φ is free.

Non-free occurrences of a variable in a formula are called *bound* occurrences. We write "$\varphi(x_1, \ldots, x_n)$" for "$\varphi$" to indicate that all variables occurring free in φ are among the (distinct, in the default case) variables $x_1, \ldots, x_n$.

Abbreviations:

$$
\begin{array}{rcl}
(\varphi \vee \psi) & \text{for} & \neg(\neg\varphi \ \wedge \ \neg\psi) \\
(\varphi \rightarrow \psi) & \text{for} & (\neg\varphi \ \vee \ \psi) \\
(\varphi \leftrightarrow \psi) & \text{for} & ((\varphi \rightarrow \psi) \ \wedge \ (\psi \rightarrow \varphi)) \\
(\forall x) & \text{for} & \neg(\exists x)\neg \\
x \neq y & \text{for} & \neg\, x = y \\
x \notin y & \text{for} & \neg\, x \in y
\end{array}
$$

We often omit parentheses, and we often write "x," "y," etc., when when we should be writing "v" with subscripts.

The Zermelo–Fraenkel (ZFC) Axioms. Below we list the formal ZFC axioms. Following each axiom, we give in parentheses an informal version of it. Our official axioms are the formal ones.

For all the axioms other those of the Comprehension and Replacement Schema, let us use the following scheme of "abbreviation":

$$
\begin{array}{lllllllllll}
x & \text{for} & v_1 & z & \text{for} & v_3 & w & \text{for} & v_5 & y_2 & \text{for} \ v_7 \\
y & \text{for} & v_2 & u & \text{for} & v_4 & y_1 & \text{for} & v_6 & &
\end{array}
$$

For the two schemata, the variables are arbitrary. I.e., there is an instance of Comprehension for each formula φ and sequence $x, y, z, w_1, \ldots, w_n$ of distinct variables that contains all variables occurring free in φ plus the variable y that does not so occur.

Axiom of Set Existence:

$$(\exists x)\, x = x$$

(There is a set.)

Axiom of Extensionality:

$$(\forall x)(\forall y)\,((\forall z)(z \in x \leftrightarrow z \in y) \rightarrow x = y)$$

(Sets that have the same members are identical.)

Axiom of Foundation:

$$(\forall x)\,((\exists y)\, y \in x \rightarrow (\exists y)(y \in x \,\wedge\, (\forall z)(z \notin x \,\vee\, z \notin y)))$$

(Every non-empty set x has a member that has no members in common with x.)

Axiom Schema of Comprehension: For each formula $\varphi(x, z, w_1, \ldots, w_n)$,

$$(\forall w_1) \cdots (\forall w_n)(\forall z)(\exists y)(\forall x)\,(x \in y \leftrightarrow (x \in z \,\wedge\, \varphi))$$

(For any set z and any property P, there is a set whose members are those members of z that have property P.)

Axiom of Pairing:

$$(\forall x)(\forall y)(\exists z)(x \in z \,\wedge\, y \in z)$$

(For any sets x and y, there is a set to which both x and y belong, i.e., of which they are both members.)

Axiom of Union:

$$(\forall x)(\exists y)(\forall z)(\forall w)\,((w \in z \,\wedge\, z \in x) \rightarrow w \in y)$$

(For any set x, there is a set to which all members of members of x belong.)

The axioms of Pairing, Union, and Comprehension give us some operations on sets. For any x and y, $\{x, y\}$ is the set whose members are exactly x and y. (It exists by Pairing and Comprehension.) Let $\{x \mid \varphi(x, \ldots)\}$ be the set of all x such that $\varphi(x, \ldots)$ holds, if this is a set. For any set x,

$$\mathcal{U}(x) = \{z \mid (\exists y)(z \in y \,\wedge\, y \in x)\}.$$

($\mathcal{U}(x)$ exists by Union and Comprehension.) For any sets x and y, $x \cup y$ is the set $\mathcal{U}(\{x, y\})$. For any sets $x_1, \ldots, x_n$, $\{x_1, \ldots, x_n\}$ is the set whose members are exactly $x_1, \ldots, x_n$. (To see that this set exists, note that $\{x\} = \{x, x\}$ for any set x and that $\{x_1, \ldots, x_{m+1}\} = \{x_1, \ldots, x_m\} \cup \{x_{m+1}\}$ for $0 \leq m < n$.)

In the statement of the next axiom, "$(\exists! y)$" is short for the obvious way of expressing "there is exactly one y."

Axiom Schema of Replacement: For each formula $\varphi(x, y, z, w_1, \ldots, w_n)$,

$$(\forall w_1) \cdots (\forall w_n)(\forall z)\,((\forall x)(x \in z \rightarrow (\exists! y)\,\varphi) \\ \rightarrow (\exists u)(\forall x)(x \in z \rightarrow (\exists y)(y \in u \wedge \varphi)))$$

(For any set z and any relation R, if each member x of z bears R to at exactly one set y_x, then there is a set to which all these y_x belong.)

Remark. by Comprehension, Replacement can be strengthened to give

$$(\forall w_1) \cdots (\forall w_n)(\forall z)\,((\forall x)(x \in z \rightarrow (\exists! y)\,\varphi) \\ \rightarrow (\exists u)(\forall y)(y \in u \leftrightarrow (\exists x)(x \in z \wedge \varphi)))\,.$$

Define $\mathcal{S}(x) = x \cup \{x\}$. Note that $\emptyset$ exists by Set Existence and Comprehension.

Axiom of Infinity:

$$(\exists x)\,(\emptyset \in x \wedge (\forall y)(y \in x \rightarrow \mathcal{S}(y) \in x))$$

(There is a set that has the empty set as a member and is closed under the operation $\mathcal{S}$.)

Let "$z \subseteq x$" abbreviate "$(\forall w)(w \in z \rightarrow w \in x)$."

Axiom of Power Set.

$$(\forall x)(\exists y)(\forall z)(z \subseteq x \rightarrow z \in y)$$

(For any set x, there is a set to which all subsets of x belong.)

Let $\mathcal{P}(x) = \{z \mid z \subseteq x\}$. (It exists by Power Set and Comprehension.) Let $x \cap y = \{z \mid z \in x \wedge z \in y\}$. (It exists by Comprehension.)

Axiom of Choice:

$$(\forall x)\,((\forall y_1)(\forall y_2)\,((y_1 \in x \,\wedge\, y_2 \in x) \to (y_1 \neq \emptyset \,\wedge\, (y_1 = y_2 \,\vee\, y_1 \cap y_2 = \emptyset))) \\ \to\; (\exists z)(\forall y)(y \in x \to (\exists! w)\, w \in y \cap z)))$$

(If x is a set of non-empty sets no two of which have any members in common, then there is a set that has exactly on member in common with each member of x.)

Remark. For all the axioms except Comprehension and Replacement, the formal and informal versions are equivalent. But the formal Comprehension and Replacement Schemata are *prima facie* weaker than the informal versions. The formal schemata apply, not to arbitrary properties and relations, but only to properties and relations characterizable by formulas of the formal language. (Warning: We shall later use the word "relation" in a precise technical sense quite different from the intuitive way we used the word in stating the informal version of Replacement.)

Justifications of the axioms. The ZFC axioms are supposed to be true of the iterative concept of set. Following is an axiom-by-axiom attempt to explain why.

Set Existence. $\emptyset$ belongs to V_1.

Extensionality. It follows from the notion of identity for collections.

Foundation. Assume $x \neq \emptyset$. Let w be the collection of all sets formed before any member of x is formed. Some member of x is formed at some stage s. Since w is a subcollection of V_s, clause (f) of the iterative concept implies that w is formed as a set at some stage s_1 no later than s. No $y \in x$ can be formed at a stage s_2 before s_1, for then w would be a subcollection of V_{s_2} and so would be formed at or before s_2. If no $y \in x$ were formed at s_1, then V_{s_1+1} would be included in w, and so w would belong to itself, an impossibility. Any $y \in x$ formed at s_1 has the right properties.

Comprehension. The desired y is a subcollection of z and so of V_s, where z is formed at s.

Pairing. If x and y are formed at or before s, then they belong to V_{s+1}, which therefore works as z.

Union. If x is formed at s, then all members of x, and so all members of members of x, belong to V_s. Hence V_s works as y.

Replacement. For each $x \in z$, let s_x be the stage at which the unique y such that $\varphi(x, y, z, w_1, \ldots, w_n)$ is formed. The collection of all these s_x is no larger than the set z, so "absolute infinity" demands that there be a stage s later than all the s_x. Then V_s works as u.

Infinity. By absolute infinity, there is an infinite stage s. Let x be the collection of all y in V_s that are formed at finite stages. Then x has the required properties and is formed at or before s.

Power Set. If x is formed at s and if $z \subseteq x$, then $z \subseteq V_s$ and so $z \in V_{s+1}$. Thus V_{s+1} works for y.

Choice. If x is formed at s, then we are looking for a z that might as well be a subcollection of $\mathcal{U}(x) \subseteq V_s$. What we have to convince ourselves is that such a subcollection exists.

The *ordered pair* $\langle x, y \rangle$ of sets x and y is $\{\{x\}, \{x, y\}\}$. Note that

$$\langle x, y \rangle = \langle z, w \rangle \leftrightarrow (x = z \wedge y = w).$$

Exercise 1.1. Write a formula of the formal language expressing the statement that $w = \langle x, y \rangle$.

The *Cartesian product* $u \times v$ of sets u and v is $\{\langle x, y \rangle \mid x \in u \wedge y \in v\}$.

Theorem 1.1. *$u \times v$ always exists.*

Proof 1. Let $x \in u$. Then $(\forall y \in v)(\exists! w)\, w = \langle x, y \rangle$. Here, and later, we use obvious abbreviations, such as "$(\forall y \in v) \ldots$," without explicit mention. By Replacement and Comprehension, let $z_x = \{w \mid (\exists y \in v)\, w = \langle x, y \rangle\}$. Then $(\forall x \in u)(\exists! z)\, z = z_x$. (Note that there is a formula $\psi(x, z, u, v)$ expressing the statement that $z = z_x$.) By Replacement and Comprehension, let $q = \{z_x \mid x \in u\}$. The Cartesian product of u and v is $\mathcal{U}(q)$. $\square$

Proof 2. $\mathcal{P}(\mathcal{P}(u \cup v))$ exists by Power Set and Comprehension. If $x \in u$ and $y \in v$, then $\langle x, y \rangle \in \mathcal{P}(\mathcal{P}(u \cup v))$. Thus $u \times v$ exists by Comprehension. $\square$

Remark. Proof 1 used Replacement but not Power Set. Proof 2 used Power Set but not Replacement.

A *relation* is a set of ordered pairs. A *function* is a relation f such that

$$(\forall x)(\forall y_1)(\forall y_2)((\langle x, y_1 \rangle \in f \wedge \langle x, y_2 \rangle \in f) \rightarrow y_1 = y_2).$$

The definitions of a *one-one* function, the *domain* of a function, and the *range* of a function are the obvious ones. The notation $f : x \to y$ means, as usual, that f is a function whose domain is x and whose range is $\subseteq y$.

A set r is a *linear ordering* of a set x if r is a *relation in* x (i.e., $r \subseteq x \times x$) and r linearly orders x in the usual strict sense (i.e., we require that $\langle y, y \rangle \notin r$).

A relation r is *wellfounded* if

$$(\forall x)(x \neq \emptyset \to (\exists y \in x)(\forall z \in x)\, \langle z, y \rangle \notin r).$$

Example. Let u be a set. Let

$$\in \restriction u = \{\langle z, y \rangle \in u \times u \mid z \in y\}.$$

The Axiom of Foundation says that $\in \restriction u$ is wellfounded for every u.

We say that r *is a wellordering* of x if r is a linear ordering of x and r is wellfounded. We say that r *wellorders* x if r is a relation and $r \cap (x \times x)$ is a wellordering of x.

A set x is *transitive* if $\mathcal{U}(x) \subseteq x$.

An *ordinal number* is a set x such that

(1) x is transitive;

(2) $\in \restriction x$ wellorders x.

Remark. Foundation implies that (2) is equivalent with the assertion that $\in \restriction x$ linearly orders x.

Exercise 1.2. Let x and y be ordinal numbers. Show, without using Foundation, that

$$x \in y \lor y \in x \lor x = y.$$

Hint. Let $z = x \cap y$. Show that z is an ordinal number. Next show that $z \in x$ or $z = x$ and also that $z \in y$ or $z = y$. For the first of these, assume that $z \neq x$. Since $z \subseteq x$, Extensionality implies that the set $x \setminus z = \{w \in x \mid w \notin z\}$ is non-empty and so has an $\in$-least member u. Prove that z and u have the same members.

The set ω is defined as follows:

$$x \in \omega \leftrightarrow (\forall y)((\emptyset \in y \land (\forall z)(z \in y \to \mathcal{S}(z) \in y)) \to x \in y).$$

ω exists by Infinity and Comprehension. Note that

$$\emptyset \in \omega \ \wedge \ (\forall z)(z \in \omega \to \mathcal{S}(z) \in \omega).$$

The members of ω are called *natural numbers.*

Remark. In preparation for metamathematical results in 220C, we shall make note of all uses of Foundation or Choice in proving theorems, and we shall avoid using these axioms unnecessarily. In particular, we avoid using Foundation in the following proofs, although using it would simplify matters.

Theorem 1.2. *ω is a set of ordinal numbers; i.e., every natural number is an ordinal number.*

Proof. Let $y = \{n \in \omega \mid n \text{ is an ordinal number}\}$; y exists by Comprehension. It is easy to see that $\emptyset \in y$. Let $n \in \omega$. We assume that $n \in y$ and show that $\mathcal{S}(n) \in y$. This will prove that $\omega \subseteq y$, and so that $y = \omega$.

By the definition of $\mathcal{S}(n)$,

$$(\forall u)(u \in \mathcal{S}(n) \leftrightarrow (u \in n \ \vee \ u = n)).$$

Hence, for any v, $v \in \mathcal{U}(\mathcal{S}(n)) \Leftrightarrow (v \in \mathcal{U}(n) \vee v \in n) \Rightarrow$ (since n is transitive) $v \in n \Rightarrow v \in \mathcal{S}(n)$. Hence $\mathcal{S}(n)$ is transitive.

$n \notin n$, since otherwise $\in \restriction n$ is not wellfounded, indeed is not even a linear ordering of n. Moreover n does not belong to any $u \in n$, since otherwise transitivity gives $n \in n$. Thus the relation $\in \restriction \mathcal{S}(n)$ is just the wellordering $\in \restriction n$ with n stuck on at the end. It is easy to prove that $\in \restriction \mathcal{S}(n)$ is a wellordering, using the fact that $\in \restriction n$ is wellordering. $\square$

Remark. The method used to prove the last theorem is mathematical induction. To prove that every natural number has a property (such as being an ordinal number), we prove that $\emptyset$ has the property and that if $n \in \omega$ has the property then so does $\mathcal{S}(n)$. By the definition of ω, this implies that the set of all natural numbers with the property is all of ω, i.e., that every natural number has the property.

Theorem 1.3. *ω is an ordinal number.*

Proof. Let $y = \{n \in \omega \mid n \subseteq \omega\}$. To prove that ω is transitive, we must show that $y = \omega$. We use mathematical induction. Trivially $\emptyset \in y$. Suppose $n \in y$. Then $u \in \mathcal{S}(n) \Leftrightarrow (u \in n \ \vee \ u = n) \Rightarrow u \in \omega$. Hence $\mathcal{S}(n) \subseteq \omega$. But also $\mathcal{S}(n) \in \omega$, so $\mathcal{S}(n) \in y$.

Theorem 1.2 and its proof show that $\in\restriction\omega$ is irreflexive ($n \notin n$ for $n \in \omega$) and asymmetric ($m \in n \to n \notin m$ for m and n elements of ω). The fact that every member of ω is transitive implies directly that $\in\restriction\omega$ is a transitive relation ($k \in m \in n \to k \in n$ for for k, m, and n elements of ω). Exercise 1.2 and Theorem 1.2 imply that $\in\restriction\omega$ is connected ($m \in n \vee n \in m \vee m = n$ for m and n elements of ω). Thus $\in\restriction\omega$ is a linear ordering of ω.

To show that $\in\restriction\omega$ is wellfounded, we prove that each non-empty subset of ω has a $(\in\restriction\omega)$-least element. Let $v \subseteq \omega$ with $v \neq \emptyset$. Let $n \in v$. If $n \cap v = \emptyset$, then n is the $(\in\restriction\omega)$-least element of v. Suppose then that $n \cap v \neq \emptyset$. By Theorem 1.2, the set $n \cap v$ has an $(\in\restriction n)$-least element m. The transitivity of n implies that m is also the $(\in\restriction\omega)$-least element of v. $\square$

Sometimes we shall want to assert *theorem schemata* rather than simple theorems: we shall want to assert that, for every formula φ, some sentence derived from φ is a theorem. A convenient way to do this is to speak of *classes.* We shall speak of $\{x \mid \varphi(x,\ldots)\}$ as a *class* whether or not there is a set $\{x \mid \varphi(x,\ldots)\}$. When the set exists, we identify the set and the class. When the set does not exist, we call $\{x \mid \varphi(x,\ldots)\}$ a *proper class.* Lower case letters will be used only for sets. Upper case letters will be used mostly for classes.

Terms like *relation, function, domain, wellfounded*, etc. are defined for classes just as they are for sets. In class language, the Comprehension Schema says that the intersection of a class and a set is a set.

Let $V = \{x \mid x = x\}$. V is a proper class, since otherwise Comprehension would yield the self-contradictory *Russell set* $\{x \mid x \notin x\}$.

An example of a proper class relation is $\in = \{\langle x, y\rangle \mid x \in y\}$. In the hint to Exercise 1.2, we wrote "$\in$" instead of $\in\restriction x$ and $\in\restriction y$. Retroactively this notation is now explained.

Exercise 1.3. Prove that $\in$ is a proper class.

If F is a class function and A is a class, then $F\restriction A = \{\langle x, y\rangle \in F \mid x \in A\}$.

Theorem 1.4 (Schema of Definition by Recursion). *Let $F : V \to V$. There is a unique (set) $g : \omega \to V$ such that*

$$(\forall n \in \omega)\, g(n) = F(g\restriction n).$$

Proof. We first show that

$$(\forall n \in \omega)(\exists! g)(g : n \to V \ \wedge\ (\forall m \in n)\, g(m) = F(g\restriction m)).$$

For $n = \emptyset$, the empty g (i.e., $\emptyset$) works. Suppose $g : n \to V$ is the unique function with the property $(\forall m \in n)\, g(m) = F(g \restriction m)$. Let $g' = g \cup \{\langle n, F(g)\rangle\}$. Clearly $g' : \mathcal{S}(n) \to V$ and $(\forall m \in \mathcal{S}(n))\, g'(m) = F(g' \restriction m)$. If $h : \mathcal{S}(n) \to V$ satisfies $(\forall m \in \mathcal{S}(n))\, h(m) = F(h \restriction m)$, then $h \restriction n = g$ by the uniqueness property of g. But then $h(n) = F(h \restriction n) = F(g) = g'(n)$, and so $h = g'$. Our conclusion follows by induction.

By Replacement and Comprehension, let

$$z = \{y \mid (\exists n \in \omega)(y : n \to V \ \wedge \ (\forall m \in n)\, y(m) = F(y \restriction m))\}.$$

Suppose y_1 and y_2 belong to z. Let $y_1 : n_1 \to V$ and $y_2 : n_2 \to V$. If $n_1 = n_2$ then the uniqueness part of the assertion proved in the last paragraph gives $y_1 = y_2$. If $n_1 \in n_2$ then uniqueness gives $y_1 = y_2 \restriction n_1$; if $n_2 \in n_1$ then uniqueness gives $y_2 = y_1 \restriction n_1$. Thus $y_1 \subseteq y_2$ or $y_2 \subseteq y_1$. Let $g = \mathcal{U}(z)$. It is easy to see that g is a function and that domain $(g) \subseteq \omega$. To see that $\omega \subseteq$ domain (g), use the existence part of the assertion of the last paragraph to get, for each $n \in \omega$, a $y \in z$ with $y : \mathcal{S}(n) \to V$. It is easy to see that $(\forall n \in \omega)\, g(n) = F(g \restriction n)$. For uniqueness, assume that $(\forall n \in \omega)\, h(n) = F(h \restriction n)$. For each $n \in \omega$, $g \restriction \mathcal{S}(n) = h \restriction \mathcal{S}(\backslash)$, and so $g(n) = h(n)$. $\square$

Remark. We needed Replacement only to get that g is a set (rather than a proper class).

Theorem 1.5. $(\forall x)(\exists y)(y \text{ is transitive } \wedge\ x \subseteq y)$.

Proof. Define $F : V \to V$ by

$$F(z) = u \ \leftrightarrow \ \begin{cases} z \text{ is not a function and } u = \emptyset \\ \text{or } z \text{ is a function and } u = x \cup \mathcal{U}(\mathcal{U}(\text{range}\,(z))). \end{cases}$$

Let g be given by Theorem 1.4. Let $y = \mathcal{U}(\text{range}\,(g))$. Suppose $v \in y$. Then $v \in g(n)$ for some $n \in \omega$. Hence $v \in \mathcal{U}(\text{range}\,(g \restriction \mathcal{S}(n)))$. Therefore

$$v \subseteq \mathcal{U}(\mathcal{U}(\text{range}\,(g \restriction \mathcal{S}(n)))) \subseteq F(g \restriction \mathcal{S}(n)) = g(\mathcal{S}(n)) \subseteq y.$$

Since $x = g(0)$, it follows that $x \subseteq y$. $\square$

For any class A, let

$$\bigcap A = \{z \mid (\forall y \in A)\, z \in y\}.$$

Comprehension gives that $\bigcap A$ is a set if A is non-empty. Note that $\omega = \bigcap\{y \mid \emptyset \in y \wedge (\forall z \in y)\, \mathcal{S}(z) \in y\}$. The operation dual, in a natural sense, to $\bigcap$ is the operation $\mathcal{U}$. We shall hence sometimes write $\bigcup x$ for $\mathcal{U}(x)$.

For any set x let

$$\operatorname{trcl}(x) = \bigcap\{y \mid y \text{ is transitive } \wedge x \subseteq y\}.$$

Theorem 1.5 implies that $\operatorname{trcl}(x)$, the *transitive closure* of x, is always a set.

Theorem 1.6. *Let*

$$\mathrm{ON} = \{x \mid x \text{ is an ordinal number}\}.$$

The (class) relation $\in \restriction \mathrm{ON}$ *is a wellordering of* ON. *Indeed* $\in \restriction \mathrm{ON}$ *is well-founded in the strong sense that every non-empty subclass of* ON *has an* $\in$*-minimal element. Furthermore* ON *is transitive.*

Proof. The proofs that $\in \restriction \mathrm{ON}$ is irreflexive, asymmetric, transitive, and connected are just like the corresponding parts of the proof of of Theorem 1.3.

Suppose that $A \subseteq \mathrm{ON}$ is a non-empty class. Let $x \in A$. If $x \cap A = \emptyset$, then we are done. Otherwise apply the fact that $x \in \mathrm{ON}$ to $x \cap A$. This gives a $y \in x \cap A$ with $y \cap x \cap A = \emptyset$. If $z \in y \cap A$ then $z \in y \in x \in \mathrm{ON}$, and so $z \in x$.

To prove that ON is transitive, suppose $x \in y \in \mathrm{ON}$. By the transitivity of y, we have that $x \subseteq y$. The fact that $\in \restriction x$ is a wellordering thus follows easily from the fact that $\in \restriction y$ is a wellordering. To show that x is transitive, and so that x is an ordinal number, let $z \in w \in x$. We have that w, and hence z, belongs to y. Since $\in \restriction y$ is a transitive relation, we get that $z \in x$. $\square$

When we talk of $\emptyset$ in its role as an ordinal number, we shall call it 0. We denote $\in \restriction \mathrm{ON}$ by $<$. For ordinals α and β, we write the natural $\alpha < \beta$ to mean that $\langle \alpha, \beta \rangle \in <$, i.e., that $\alpha \in \beta$.

Exercise 1.4. Show, for any ordinal number α, that $\mathcal{S}(\alpha)$ is the immediate successor of α with respect to $<$.

Exercise 1.5. Let x be any set of ordinal numbers. Prove that $\mathcal{U}(x)$ is an ordinal number.

Theorem 1.6 makes possible proof by *transfinite induction.* If we want to show that all ordinal numbers have some property expressed by a formula φ, it is enough to show that, for every ordinal number α,

$$(\forall \beta < \alpha)\, \varphi(\beta, \ldots) \;\rightarrow\; \varphi(\alpha, \ldots).$$

For then Theorem 1.6 implies that the class of $\alpha \in \mathrm{ON}$ such that $\neg\varphi(\alpha, \ldots)$ cannot be non-empty. The following theorem gives us a useful division into cases when we are using transfinite induction.

Theorem 1.7. *If α is an ordinal number, then one of the following holds:*

(1) $(\exists \beta < \alpha)\, \alpha = \mathcal{S}(\beta)$*;*

(2) $\alpha = \mathcal{U}(\alpha)$*.*

Proof. Let α be an ordinal number, and assume that (1) fails. Since $\mathcal{U}(\alpha) \subseteq \alpha$ for any ordinal α, we need only show that $\alpha \subseteq \mathcal{U}(\alpha)$. Let $\beta \in \alpha$. By Exercise 1.4, $\mathcal{S}(\beta)$ is an ordinal number $\leq \alpha$. Since (1) fails, we must have $\mathcal{S}(\beta) < \alpha$. But then $\beta \in \mathcal{S}(\beta) \in \alpha$, so $\beta \in \mathcal{U}(\alpha)$. $\square$

Ordinals satisfying (1) are called *successor* ordinals. Non-zero ordinals satisfying (2) are called *limit* ordinals.

Theorem 1.8 (Schema of Definition by Transfinite Recursion). *Let $F : V \rightarrow V$. There is a (unique) $G : \mathrm{ON} \rightarrow V$ such that*

$$(\forall \alpha \in \mathrm{ON})\, G(\alpha) = F(G \restriction \alpha).$$

Proof. We first show that

$$(\forall \alpha \in \mathrm{ON})(\exists! g)(g : \alpha \rightarrow V \;\wedge\; (\forall \beta < \alpha)\, g(\beta) = F(G \restriction \beta)).$$

We argue by transfinite induction. Let α be an ordinal and assume that the statement holds for all smaller ordinals. The case $\alpha = 0$ is trivial. If $\alpha = \mathcal{S}(\beta)$ for some ordinal β, then we argue as in the proof of Theorem 1.4. If α is a limit ordinal, then we use Replacement as for the special case $\alpha = \omega$ in the last part of the proof of Theorem 1.4 to get a z that is the set of all g' that work for ordinals $\beta < \alpha$. We let $g = \mathcal{U}(z)$.

Let

$$G = \mathcal{U}(\{g \mid (\exists \alpha \in \mathrm{ON})(g : \alpha \rightarrow V \;\wedge\; (\forall \beta < \alpha)\, g(\beta) = F(g \restriction \beta))\}.$$

It is easy to check that G, and only G, has the required property. $\square$

Remark. Note that the proof gives an explicit definition of G from a definition of F. Thus the theorem really is a theorem schema, and the quantification over proper classes in its statement could be avoided.

Theorem 1.9. *There is a unique* $\mathbf{V} : \mathrm{ON} \to V$ *such that (where we write* V_α *for* $\mathbf{V}(\alpha)$*)*

(a) $V_0 = \emptyset$*;*

(b) $V_{\mathcal{S}(\alpha)} = \mathcal{P}(V_\alpha)$*;*

(c) $V_\lambda = \mathcal{U}(\{V_\alpha \mid \alpha < \lambda\})$ *if* λ *is a limit ordinal.*

Proof. Let $F(x) = \emptyset$ if $x = \emptyset$ or x is not a function whose domain is an ordinal number. If α an ordinal and $x : \mathcal{S}(\alpha) \to V$, then let $F(x) = \mathcal{P}(x(\alpha))$. If λ is a limit ordinal and $x : \lambda \to V$, let $F(x) = \mathcal{U}(\text{range}\,(x))$. The desired function is given by Theorem 1.8. $\square$

Exercise 1.6. Show that $\alpha < \beta \to V_\alpha \subseteq V_\beta$.

Theorem 1.10. (Uses Foundation) $(\forall x)(\exists \alpha)\, x \in V_\alpha$.

Proof. Suppose x belongs to no V_α. Let

$$z = \{u \in \mathrm{trcl}\,(x) \cup \{x\} \mid (\forall \alpha \in \mathrm{ON})\, u \notin V_\alpha\}.$$

Since $z \neq \emptyset$, Foundation gives a $u \in z$ such that $u \cap z = \emptyset$. Every member of u belongs to $\mathrm{trcl}\,(x)$, and so every member of u belongs to some V_α. For $y \in u$, let α_y be the least α such that $y \in V_\alpha$. By Replacement and Comprehension, let $\alpha = \mathcal{U}(\{\alpha_y \mid y \in u\})$. By Exercise 1.5, $\alpha \in \mathrm{ON}$. By Exercise 1.6, $u \subseteq V_\alpha$. This gives the contradiction that $u \in V_{\mathcal{S}(\alpha)}$. $\square$

By tranfinite recursion, one can define addition, multiplication, and exponentiation of ordinal numbers as follows:

$$\begin{aligned}
\alpha + 0 &= \alpha\,; \\
\alpha + \mathcal{S}(\beta) &= \mathcal{S}(\alpha + \beta)\,; \\
\alpha + \lambda &= \mathcal{U}(\{\alpha + \beta \mid \beta < \lambda\}) \text{ if } \lambda \text{ is a limit ordinal.}
\end{aligned}$$

$$\begin{aligned}
\alpha \cdot 0 &= 0\,; \\
\alpha \cdot \mathcal{S}(\beta) &= \alpha \cdot \beta + \alpha\,; \\
\alpha + \lambda &= \mathcal{U}(\{\alpha \cdot \beta \mid \beta < \lambda\}) \text{ if } \lambda \text{ is a limit ordinal.}
\end{aligned}$$

$$\begin{aligned} \alpha^0 &= 1 \ \ (= \mathcal{S}(0))\,; \\ \alpha^{\mathcal{S}(\beta)} &= \alpha^\beta \cdot \alpha\,; \\ \alpha^\lambda &= \mathcal{U}(\{\alpha^\beta \mid \beta < \lambda\}) \text{ if } \lambda \text{ is a limit ordinal.} \end{aligned}$$

The way this is done is as follows: Consider the definition of $+$. We can define a function $F : \mathrm{ON} \times V \to V$, so that, e.g., if α and β are ordinals and $x : \mathcal{S}(\beta) \to V$, then $F(\langle \alpha, x \rangle) = \mathcal{S}(x(\beta))$. If we define $F_\alpha : V \to V$ by $F_\alpha(x) = F(\langle \alpha, x \rangle)$, then Theorem 1.8 applied to F_α gives a function $+_\alpha : \mathrm{ON} \to \mathrm{ON}$. Since the proof of Theorem 1.8 gives us a definition of the $+_\alpha$ from the parameter α, we get an explicit definition of $+$.

Note that $\alpha + 1 = \mathcal{S}(\alpha)$ for every ordinal α. We shall often write $\alpha + 1$ instead of $\mathcal{S}(\alpha)$. For the rest of this section, however, we shall continue to write $\mathcal{S}(\alpha)$ in order to avoid confusion with the different kind of addition that we shall shortly define.

We now turn to the subject of cardinal numbers. If x and y are sets, let us say that $x \preceq y$ if there is a one-one $f : x \to y$. By $x \approx y$ we mean that there is a one-one onto $f : x \to y$.

Theorem 1.11 (Schröder–Bernstein Theorem). *If $x \preceq y$ and $y \preceq x$ then $x \approx y$.*

Proof. Let $f : x \to y$ and $g : y \to x$ be one-one. Using Theorem 1.4, define $h : x \times \omega \to x$ by

$$\begin{aligned} h(z, 0) &= z\,; \\ h(z, \mathcal{S}(n)) &= g(f(h(z, n)))\,. \end{aligned}$$

Let

$$u = \{z \in x \mid (\exists v \in x)(\exists n \in \omega)(h(v, n) = z \wedge v \notin \mathrm{range}\,(g))\}.$$

Note that if $z \notin u$ then $z \in \mathrm{range}\,(g)$. Let $k : x \to y$ be given by

$$k(z) = \begin{cases} f(z) & \text{if } z \in u; \\ g^{-1}(z) & \text{if } z \notin u. \end{cases}$$

(If r is any relation, $r^{-1} = \{\langle w, w' \rangle \mid \langle w', w \rangle \in r\}$. Since g is a one-one function, we have that $g^{-1} : \mathrm{range}\,(g) \to y$.)

To see that k is one-one, assume that $k(z_1) = k(z_2)$. Exchanging z_1 and z_2 if necessary, we may assume that either $z_1 = z_2$ or else $z_1 \in u$ and $z_2 \notin u$. Assume for a contradiction that the latter is the case. Then $f(z_1) = g^{-1}(z_2)$, and so $g(f(z_1)) = z_2$. Let v and n witness that $z_1 \in u$. Since $h(v,n) = z_1$, we get that $g(f(h(v,n))) = g(f(z_1)) = z_2$. This means that $h(v, \mathcal{S}(n)) = z_2$, contradicting the fact that $z_2 \notin u$.

Assume that $z \in y \setminus \mathrm{range}\,(k)$. Then $g(z) \in u$, since otherwise $k(g(z)) = g^{-1}(g(z)) = z$. Let v and n witness that $g(z) \in u$. Obviously $n \neq 0$. Thus $n = \mathcal{S}(m)$ for some m. We have then that $g(z) = h(v, \mathcal{S}(m)) = g(f(h(v,m))$. Hence $z = f(h(v,m))$. But $h(v,m) \in u$, and so we get the contradiction that

$$k(h(v,m)) = f(h(v,m)) = z.$$

$\square$

A *cardinal number* is an ordinal number α such that $(\forall \beta < \alpha)\, \beta \not\approx \alpha$.

Theorem 1.12. *Every natural number is a cardinal number. ω is a cardinal number.*

Proof. For the first assertion, we show that

$$(*) \qquad (\forall n \in \omega)(\forall f)((f : n \to n \,\wedge\, f \text{ one-one}) \,\to f \text{ onto}).$$

The case $n = 0$ is trivial. Let $f : \mathcal{S}(n) \to \mathcal{S}(n)$ be one-one. We must have that $n \in \mathrm{range}\,(f)$, since otherwise $f \restriction n : n \to n$ is not onto. Let $a = f(n)$ and let $f(b) = n$. Define $g : n \to n$ by

$$g(m) = \begin{cases} f(m) & \text{if } m \neq b; \\ a & \text{if } m = b. \end{cases}$$

By the induction hypothesis, $\mathrm{range}\,(g) = n$. Thus

$$\mathrm{range}\,(f) = \{n\} \cup \mathrm{range}\,(g) = \mathcal{S}(n).$$

For the second assertion, note that if $n \in \omega$ and $f : \omega \to n$ is one-one, then $f \restriction \mathcal{S}(n) : \mathcal{S}(n) \to n$ contradicts $(*)$. $\square$

Theorem 1.13. *Let $\alpha \in \mathrm{ON} \setminus \omega$. Then $\mathcal{S}(\alpha)$ is not a cardinal number.*

Proof. Define $f : \mathcal{S}(\alpha) \to \alpha$ by

$$f(\beta) = \begin{cases} \mathcal{S}(n) & \text{if } n < \omega; \\ \beta & \text{if } \omega \le \beta < \alpha; \\ 0 & \text{if } \beta = \alpha. \end{cases}$$

$\square$

Let card(x) $(= |x|)$ be the least cardinal number κ such that $x \approx \kappa$, if it exists. Note that card(α) exists for all ordinals α. The following theorem implies that that card(x) exists if x can be wellordered, i.e., if there is a wellordering of x.

Theorem 1.14. *Let r be a wellordering of x. Then there is an ordinal number α such that $\langle x, r\rangle$ is isomorphic to $\langle \alpha, \in \restriction \alpha\rangle$, i.e., there is a one-one onto $f : \alpha \to x$ such that*

$$\beta < \gamma < \alpha \to \langle f(\beta), f(\gamma)\rangle \in r.$$

Furthermore, both α and the isomorphism f are unique.

Proof. Note that α and f must satisfy

$$(\forall \beta < \alpha)\, f(\beta) \text{ is the } r\text{-least element of } x \setminus \text{range}\,(f \restriction \beta).$$

Define $F : V \to V$ as follows. Let $F(z)$ be the r-least element of $x \setminus \text{range}\,(z)$ if $(\exists \beta \in \text{ON})(\, z : \beta \to x \wedge \text{range}\,(z) \neq x)$, and let $F(z) = \emptyset$ otherwise. Let G be given by Theorem 1.8.

For each ordinal β, if range$(G \restriction \beta) \subsetneq x$ then $G(\beta) \in x \setminus \text{range}\,(G \restriction \beta)$.

Suppose that range$(G \restriction \beta) \subsetneq x$ for every ordinal β. Then $G : \text{ON} \to x$ and G is one-one. By Replacement (and Comprehension), we get that ON is a set. By Theorem 1.6, this implies that ON $\in$ ON, which contradicts Theorem 1.6.

Thus there is a $\beta \in \text{ON}$ such that range$(G \restriction \beta)$ is not a proper subset of x. Let α be the least such ordinal. If α is a limit ordinal, then range$(G \restriction \alpha) \subseteq x$ and so range$(G \restriction \alpha) = x$. This follows also if $\alpha = \mathcal{S}(\beta)$, since $G(\beta) \in x$. In both cases is it easy to see that $G \restriction \alpha$ is the desired isomorphism. $\square$

For cardinal numbers κ and δ, we define the *cardinal sum* $\kappa + \delta$ of κ and δ by

$$\kappa + \delta = \text{card}\,(\{0\} \times \kappa) \cup (\{1\} \times \delta)),$$

if it exists. Our notation is ambiguous; we use the same symbol "+" both for the cardinal sum and for the *ordinal sum*, i.e., for the + operation on ordinal numbers defined on page 13. For the rest of this section, we shall avoid confusion by writing $\alpha +_{\mathrm{ON}} \beta$ for the ordinal sum of α and β.

Theorem 1.15. *(a) For all cardinal numbers κ and δ, $\kappa + \delta$ exists.*

(b) For m and $n \in \omega$, $m + n = m +_{\mathrm{ON}} n \in \omega$.

(c) If either of κ and δ does not belong to ω, then $\kappa + \delta = \max\{\kappa, \delta\}$ $(= \mathcal{U}(\{\kappa, \delta\}))$.

Proof. (a) Define an ordering $r_{\kappa,\delta}$ of $(\{0\} \times \kappa) \cup (\{1\} \times \delta)$ by placing $\langle i, \alpha \rangle$ before $\langle j, \beta \rangle$ if and only if

$$\alpha < \beta \vee (\alpha = \beta \wedge i < j).$$

It is easy to show that $r_{\kappa,\delta}$ is a wellordering. Let $f_{\kappa,\delta} : \alpha_{\kappa,\delta} \to (\{0\} \times \kappa) \cup (\{1\} \times \delta)$ be given by Theorem 1.14. Then $\kappa + \delta = \mathrm{card}\,(\alpha_{\kappa,\delta})$.

(b) For fixed $m \in \omega$, we prove by induction on n that $m +_{\mathrm{ON}} n \in \omega$ and $m +_{\mathrm{ON}} n \approx (\{0\} \times m) \cup (\{1\} \times n)$. By definition, $m +_{\mathrm{ON}} 0 = m \in \omega$, and we can define a one-one onto $f : m \to \{0\} \times m$ by setting $f(k) = \langle 0, k \rangle$ for each $k < m$. Assume that $m +_{\mathrm{ON}} n \in \omega$ and that $f : m +_{\mathrm{ON}} n \to (\{0\} \times m) \cup (\{1\} \times n)$ is one-one and onto. Then $m +_{\mathrm{ON}} \mathcal{S}(n) = \mathcal{S}(m +_{\mathrm{ON}} n) \in \omega$. Let

$$f' = f \cup \{\langle m +_{\mathrm{ON}} n, \langle 1, n \rangle \rangle\}.$$

It is easy to see that $f' : m +_{\mathrm{ON}} \mathcal{S}(n) \to (\{0\} \times m) \cup (\{1\} \times \mathcal{S}(n))$ is one-one and onto.

(c) It is enough to prove that $\kappa + \kappa = \kappa$ for every cardinal number $\kappa \notin \omega$. Assume that this is false, and let κ be the $<$-least counterexample. Note that $r_{\kappa,\kappa}$ is a wellordering of $2 \times \kappa$, where $2 = \{0, 1\}$. We have that

$$\kappa < \kappa + \kappa \leq \alpha_{\kappa,\kappa}.$$

Let $f_{\kappa,\kappa}(\kappa) = \langle i, \beta \rangle$. Thus

$$\kappa \approx \{\langle j, \gamma \rangle \mid \langle j, \gamma \rangle \, r_{\kappa,\kappa} \, \langle i, \beta \rangle\} \subseteq (2 \times \beta) \cup \{\langle 0, \beta \rangle\} \approx \mathcal{S}(\mathrm{card}\,(\beta) + \mathrm{card}\,(\beta)).$$

If $\beta \in \omega$, then we would also have $\kappa \in \omega$. Hence the minimality of κ gives that $\kappa \preceq \mathcal{S}(\mathrm{card}\,(\beta))$, and Theorems 1.11 and 1.13 then give the contradiction that $\kappa \approx \mathrm{card}\,(\beta)$. $\square$

For cardinal numbers κ and δ, we define the *cardinal product* $\kappa \cdot \delta$ of κ and δ by

$$\kappa \cdot \delta = \mathrm{card}\,(\kappa \times \delta),$$

if it exists. Our notation is once more ambiguous, so for the rest of this section we shall write $\cdot_{\mathrm{ON}}$ for the *ordinal product* defined on page 13.

Theorem 1.16. *(a) For all cardinal numbers κ and δ, $\kappa \cdot \delta$ exists.*

(b) For m and $n \in \omega$, $m \cdot n = m \cdot_{\mathrm{ON}} n \in \omega$.

(c) If either of κ and δ does not belong to ω and neither of κ and δ is 0, then $\kappa \cdot \delta = \max\{\kappa, \delta\}$.

Exercise 1.7. Prove Theorem 1.16.

Hint: (a) Define an ordering $s_{\kappa,\delta}$ of $\kappa \times \delta$ as follows:

$$\langle \alpha, \beta \rangle \, s_{\kappa,\delta} \, \langle \alpha', \beta' \rangle \leftrightarrow \begin{cases} \max\{\alpha, \beta\} < \max\{\alpha', \beta'\} \;\; \vee \\ \max\{\alpha, \beta\} = \max\{\alpha', \beta'\} \wedge \alpha < \alpha' \;\; \vee \\ \max\{\alpha, \beta\} = \max\{\alpha', \beta'\} \wedge \alpha = \alpha' \;\wedge\; \beta < \beta'. \end{cases}$$

Show that $s_{\kappa,\delta}$ is a wellordering. Let $f^*_{\kappa,\delta} : \alpha^*_{\kappa,\delta} \to \kappa \times \delta$ be given by Theorem 1.14. Then $\kappa \cdot \delta = \mathrm{card}\,(\alpha^*_{\kappa,\delta})$.

(b) For fixed $m \in \omega$, prove by induction that, for all $n \in \omega$, $m \cdot_{\mathrm{ON}} n \in \omega$ and $m \cdot_{\mathrm{ON}} n \approx m \times n$. The case $n = 0$ is trivial. Assume that $m \cdot_{\mathrm{ON}} n \in \omega$ and that $f : m \cdot_{\mathrm{ON}} n \to m \times n$ is one-one and onto. Then $m \cdot_{\mathrm{ON}} \mathcal{S}(n) = m \cdot_{\mathrm{ON}} n +_{\mathrm{ON}} m \in \omega$. Let

$$f' = f \cup \{\langle m \cdot_{\mathrm{ON}} n + k, \langle k, n \rangle \rangle \mid k < m\}.$$

Show that $f' : m \cdot_{\mathrm{ON}} \mathcal{S}(n) \to m \times \mathcal{S}(n)$ is one-one and onto.

(c) It is enough to prove that $\kappa \cdot \kappa = \kappa$ for every cardinal number $\kappa \notin \omega$. Assume that this is false, and let κ be the $<$-least counterexample. Let $f^*_{\kappa,\kappa} : \alpha^*_{\kappa,\kappa} \to \kappa \times \kappa$ be defined as in the hint for part (a). Then

$$\kappa < \kappa \cdot \kappa \leq \alpha^*_{\kappa,\kappa}.$$

Let $\langle \alpha, \beta \rangle = f^*_{\kappa,\kappa}(\kappa)$. Let $\rho = \max\{\alpha, \beta\}$. Use the definition of $s_{\kappa,\kappa}$, the minimality of κ, and Theorem 1.15 to deduce the contradiction that $\kappa \approx \mathrm{card}\,(\rho) \leq \rho < \kappa$.

For sets x and y, let ${}^x y = \{f \mid f : x \to y\}$. (Note that ${}^x y$ is contained in the set $\mathcal{P}(x \times y)$.) Since we to not have a convenient special notation for the *ordinal exponentiaton* defined on page 14, we defer defining cardinal exponentiation until after the next theorem, which concerns ordinal exponentiation.

Theorem 1.17. *For m and $n \in \omega$, ${}^m n \approx n^m \in \omega$, where n^m is as defined on page 14.*

Proof. Fix $n \in \omega$. For the case $m = 0$, note that ${}^0 n = \{\emptyset\} = 1 = n^0$. Assume that $n^m \in \omega$ and that $n^m \approx {}^m n$. Then $n^{\mathcal{S}(m)} = n^m \cdot_{\mathrm{ON}} n \in \omega$. Moreover

$$n^m \cdot_{\mathrm{ON}} n = n^m \cdot n \approx n^m \times n \approx {}^m n \times n \approx {}^{\mathcal{S}(m)} n.$$

(For the last $\approx$, define a one-one onto f by setting $f(\langle g, k\rangle) = g \cup \{\langle m, k\rangle\}$ for $g : m \to n$ and $k < n$.) $\square$

We now define *cardinal exponentiation* by setting $\kappa^\lambda = \mathrm{card}\,({}^\lambda \kappa)$, if it exists, for cardinal numbers κ and λ. We shall make no more use of ordinal exponentiation in this section.

Theorem 1.18. *If $0 \neq n \in \omega$ and $\kappa \notin \omega$ is a cardinal number, then $\kappa^n = \kappa$.*

Proof. Fix a cardinal number $\kappa \notin \omega$. For $n \in \omega$, define $f_n : {}^{\mathcal{S}(n)}\kappa \to {}^n\kappa \times \kappa$ by setting $f_n(g) = \langle g \restriction n, g(n)\rangle$. The functions f_n are one-one and onto.

Clearly ${}^1\kappa \approx \kappa$. Assume that $n > 0$ and that ${}^n\kappa \approx \kappa$. Then

$${}^{\mathcal{S}(n)}\kappa \approx {}^n\kappa \times \kappa \approx \kappa \times \kappa \approx \kappa. \qquad \square$$

For ordinal numbers α and sets y, let ${}^{<\alpha}y = \{f \mid (\exists \beta < \alpha)\, f : \beta \to y\}$. For cardinal numbers κ and λ, let $\kappa^{<\lambda} = \mathrm{card}\,({}^{<\lambda}\kappa)$, if it exists.

Theorem 1.19. *If $\kappa \notin \omega$ is a cardinal number, then $\kappa^{<\omega} = \kappa$.*

Proof. The theorem is an easy consequence of Theorem 1.18 and the Axiom of Choice, but we wish to avoid the latter. Let f_n be as in the proof of Theorem 1.18. Let $h : \kappa \times \kappa \to \kappa$ be one-one and onto.

Define $g_n : {}^{\mathcal{S}(n)}\kappa \to \kappa$ and $g_n^* : {}^{\mathcal{S}(n)}\kappa \times \kappa \to \kappa \times \kappa$ simultaneously by recursion as follows. Let g_0 be given by h. Given g_n, let

$$g_n^*(\langle q, \alpha\rangle) = \langle g_n(q), \alpha\rangle.$$

Now let

$$g_{\mathcal{S}(n)} = h \circ g_n^* \circ f_{\mathcal{S}(n)},$$

where $\circ$ means composition. (It is easy to justify this method of definition via Theorem 1.4.) By induction we see that each g_n is one-one and onto.

Next define a one-one $p : \omega \times \kappa \to {}^{<\omega}\kappa$ by setting $p(n, \alpha) = {g_n}^{-1}(\alpha)$. (Here we write $p(n, \alpha)$ for $p(\langle n, \alpha\rangle)$.) Since ${}^{<\omega}\kappa = \mathrm{range}\,(p) \cup \{1\}$, we get that ${}^{<\omega}\kappa \approx (\omega \times \kappa) \cup \{1\} \approx \kappa$. $\square$

Theorem 1.20. *For every set x, $x \prec {}^{x}2$, i.e., $x \preceq {}^{x}2$ and $x \not\approx {}^{x}2$.*

Proof. Fix x. It is easy to see that ${}^{x}2 \approx \mathcal{P}(x)$. We show that $x \prec \mathcal{P}(x)$.

To show that $x \preceq \mathcal{P}(x)$ define a one-one $f : x \to \mathcal{P}(x)$ by setting $f(y) = \{y\}$ for all $y \in x$.

Suppose that $f : x \to \mathcal{P}(x)$ is onto. Let $z = \{y \in x \mid y \notin f(y)\}$. Let $z = f(y)$. Then $y \in f(y) \Leftrightarrow y \notin z \Leftrightarrow y \notin f(y)$. $\square$

Theorem 1.21. *There is no greatest cardinal number.*

Proof. Let κ be a cardinal number. Let

$$a = \{\langle x, r\rangle \mid x \subseteq \kappa \wedge r \text{ is a wellordering of } x\}.$$

For $\langle x, r\rangle \in a$, let $g(x, r)$ be the unique α such that $\langle \alpha, \in \restriction \alpha\rangle$ is isomorphic to $\langle x, r\rangle$. If α is an ordinal number and $\alpha \preceq \kappa$, then there is an $\langle x, r\rangle \in a$ with $\alpha = g(\langle x, r\rangle)$. (Let $f : \alpha \to \kappa$ be one-one; let $x = \text{range}(f)$; let $\langle f(\beta), f(\gamma)\rangle \in r \Leftrightarrow \beta < \gamma$.) Let $\delta = \mathcal{U}(\text{range}(g))$. Then $\delta \in \text{ON}$ and $\kappa \prec \delta$. Indeed, δ is the least cardinal number $> \kappa$. $\square$

For any set x such that $\text{card}\,(x)$ exists, let x^+ be the least cardinal number greater than $\text{card}\,(x)$.

By transfinite recursion define

$$\begin{aligned} \aleph_0 &= \omega\,; \\ \aleph_{\mathcal{S}(\alpha)} &= {\aleph_\alpha}^+\,; \\ \aleph_\lambda &= \bigcup\{\aleph_\beta \mid \beta < \lambda\} \text{ for limit ordinals } \lambda. \end{aligned}$$

It is easy to see that the $\aleph_\alpha$, $\alpha \in \text{ON}$, are all the cardinal numbers $\geq \omega$.

Theorem 1.22. (Uses Choice) *Every set can be wellordered.*

Proof. Fix a set x. For $y \subsetneq x$, let $a_y = \{y\} \times (x \setminus y)$. Let $u = \{a_y \mid y \subsetneq x\}$. Let v be given by Choice. Define $F : V \to V$ as follows. Let $F(z)$ be the unique w such that $\langle \text{range}\,(z), w\rangle \in v$ if $(\exists \beta \in \text{ON})(\, z : \beta \to x \wedge \text{range}\,(z) \neq x)$, and let $F(z) = \emptyset$ otherwise. Let G be given by transfinite recursion. Just as in the proof of Theorem 1.14, one can show that there is an ordinal α such that $G \restriction \alpha$ is a one-one onto function from α to x. $\square$

Corollary 1.23. (Uses Choice) *For every set x, card$\,(x)$ exists. For all cardinals κ and λ, both κ^λ and $\kappa^{<\lambda}$ are defined.*

By Theorems 1.20, we have that $2^{\aleph_\alpha} > \aleph_\alpha$ for every ordinal α. The *Continuum Hypotheses* (CH) asserts that $2^{\aleph_0} = \aleph_1$, and the *Generalized Continuum Hypothesis* (GCH) asserts that $2^{\aleph_\alpha} = \aleph_{\mathcal{S}(\alpha)}$ for all ordinals α.

2 Models, compactness, and completeness

Informally we shall consider a *language* to be a set of symbols, the union of the following:

(1) a set of *constant symbols*;

(2) for each n, $0 < n \in \omega$, a set of n*-place function symbols*;

(3) for each n, $0 < n \in \omega$, a set of n*-place relation symbols.*

Since we want to use theorems of set theory in doing model theory (and for other reasons concerning 220C), we adopt the following purely set theoretic defintion as our official one.

A *language* is a pair $\langle f, p \rangle$ where

(a) $f : \omega \to V$;

(b) $p : \omega \setminus \{0\} \to V$;

(c) $(\forall m \in \omega)(\forall n \in \omega)(f(m) \cap p(n) = \emptyset \wedge (m \neq n \to (f(m) \cap f(n) = \emptyset = p(m) \cap p(n))))$;

(d) each $f(n)$ and each $p(n)$ is disjoint from $\{2{\cdot}n \mid n \in \omega\} \cup \{1, 3, 5, 7, 9, 11\}$;

(e) no function whose domain is in $\omega \setminus \{\emptyset\}$ belongs to any $f(n)$ or $p(n)$.

If $\mathcal{L} = \langle f, p \rangle$, then $f(0)$ is the set of *constant symbols* of $\mathcal{L}$; for $n > 0$, $f(n)$ is the set of n*-place function symbols* of $\mathcal{L}$; for $n > 0$, $p(n)$ is the set of n*-place relation symbols* of $\mathcal{L}$. Clause (c) says that no symbol has two uses.

Logical symbols. The following symbols will be used with every language:

Informal	Official
$v_0, v_1, v_2, \ldots$	$0, 2, 4, \ldots$
(	1
)	3
$=$	5
$\neg$	7
$\wedge$	9
$\exists$	11

The symbols $v_0, v_1, v_2, \ldots$ (officially $0, 2, 4, \ldots$) are *variables.*

Terms. Informally we can describe the *terms* of a language $\mathcal{L}$ as constituting the smallest set such that

(i) all variables and constant symbols are terms;

(ii) if F is an n-place function symbol and $t_1, \ldots, t_n$ are terms, then the expression $F(t_1 \ldots t_n)$ is a term.

More informally, we shall often add commas for clarity: $F(t_1, \ldots, t_n)$.

Officially terms of $\mathcal{L}$ are finite sequences of symbols, where a *finite sequence* is a function whose domain is a natural number. To give the official set-theoretic definition we first define some operations on finite sequences.

If $g : m \to V$ and $h : n \to V$ are finite sequences, let $g^\frown h : m + n \to V$ be given by

$$(g^\frown h)(k) = \begin{cases} g(k) & \text{if } k < m\,; \\ h(j) & \text{if } k = m + j \text{ with } j < n\,. \end{cases}$$

If h is a finite sequence of finite sequences, we define $\operatorname{concat}(h)$, the *concatenation* of h, by recursion on $\operatorname{domain}(h)$ as follows:

$$\operatorname{concat}(h) = \begin{cases} \emptyset & \text{if } \operatorname{domain}(h) = 0\,; \\ (\operatorname{concat}(h \restriction n))^\frown h(n) & \text{if } \operatorname{domain}(h) = n + 1\,. \end{cases}$$

For finite sequences f, let $\ell\mathrm{h}(f) = \operatorname{domain}(f)$. For any a, let $\langle a \rangle$ be the unique element of ${}^1\{a\}$, i.e., let it be $\{\langle 0, a \rangle\}$.

Now let

$$\mathrm{Term}_0^{\mathcal{L}} = \{\langle a \rangle \mid a \text{ is a variable or a constant symbol}\}\,.$$

For $n \in \omega$, let $\mathrm{Term}_{n+1}^{\mathcal{L}}$ be the set of all $\operatorname{concat}(h)$ such that, for some $k \in \omega \setminus \{0\}$,

(a) $h : k + 3 \to V$;

(b) $h(0) \in {}^1(f(k))$, where $\mathcal{L} = \langle f, p \rangle$;

(c) $h(1) = \langle (\rangle$ (i.e., $h(1) = \langle 1 \rangle$);

(d) $h(k + 2) = \langle) \rangle$;

(e) $(\forall j < k)\, h(2 + j) \in \bigcup\{\mathrm{Term}_m^{\mathcal{L}} \mid m \leq n\}$.

A *term* of $\mathcal{L}$ is any member of $\bigcup\{\mathrm{Term}_n^{\mathcal{L}} \mid n \in \omega\}$.

Exercise 2.1. (a) Prove unique readability for terms. That is, show that if t is a term of a language $\mathcal{L}$ not belonging to $\mathrm{Term}_0^{\mathcal{L}}$, then there are unique $k \in \omega$ and $h : k + 3 \to V$ such that $t = \operatorname{concat}(h)$ and (a)–(e) above hold of k and h, with (e) modified by replacing "$m \leq n$" by "$m \in \omega$." You may (informally) prove the informal version of this fact.

(b) Would unique readabilty for terms still hold if we dropped the parentheses? Prove your answer.

Formulas. Informally we can describe the *formulas* of $\mathcal{L}$ as forming the the smallest set satisfying the conditions

(i) if t_1 and t_2 are terms, then $t_1 = t_2$ is a formula;

(ii) if P is a a k-place relation symbol and $t_1, \ldots, t_k$ are terms, then $P(t_1 \ldots t_k)$ is a formula;

(iii) if φ is a formula, then so is $\neg\varphi$;

(iv) if φ and ψ are formulas, then so is $(\varphi \wedge \psi)$;

(v) if φ is a formula and x is a variable, then $(\exists x)\,\varphi$ is a formula.

Officially we take formulas, like terms, to be finite sequences of symbols. We let $\mathrm{Formula}_0^{\mathcal{L}}$ be the set of all *atomic formulas*, i.e., the set of all finite sequences corresponding to clauses (i) and (ii) above. For $n \in \omega$, we let $\mathrm{Formula}_{n+1}^{\mathcal{L}}$ be the set of all the sequences gotten from $\bigcup\{\mathrm{Formula}_m^{\mathcal{L}} \mid m \leq n\}$ via clauses (iii), (iv) and (v). We omit the official definition, which is similar to that of the sets Term_n.

Exercise 2.2. (a) Prove unique readability for formulas. That is, show that every formula either is atomic or else has a unique analysis via (iii), (iv), or (v).

(b) Would unique readabilty for formulas still hold if we dropped the parentheses? Prove your answer.

Officially let us define an *occurrence* of a variable x in a formula φ to be $\langle m, \varphi\rangle$ for any $m < \ell\mathrm{h}(\varphi)$ such that $\varphi(m) = x$. Similarly define the notion of an *occurrence* of a variable in a term.

By the *complexity* of a formula φ, we mean the least n such that $\varphi \in \mathrm{Formula}_n^{\mathcal{L}}$. By recursion on complexity of formulas, we define the *free* occurrences of a variable in a formula. Every occurrence of a variable in an atomic formula is *free*. An occurrence $\langle m+1, \neg\varphi\rangle$ is *free* just in case the corresponding occurrence $\langle m, \varphi\rangle$ is free. An occurrence $\langle m+1, (\varphi\wedge\psi)\rangle$ with $m < \ell\mathrm{h}(\varphi)$ is *free* just in case $\langle m, \varphi\rangle$ is free. An occurrence $\langle \ell\mathrm{h}(\varphi) + m + 2, (\varphi \wedge \psi)\rangle$ is *free* just in case $\langle m, \psi\rangle$ is free. An occurrence $\langle 2, (\exists x)\varphi\rangle$ is not free. An occurrence $\langle m + 4, (\exists y)\varphi\rangle$ of x is free just in case $\langle m, \varphi\rangle$ is free and x and y are different variables.

Models. A *model* $\mathfrak{A}$ for a language $\mathcal{L}$ is a an ordered pair consisting of (a) a non-empty set $A = |\mathfrak{A}|$, the *universe* or *domain* of the model, and (b) a function assigning

(1) to each constant symbol c, an element $c_{\mathfrak{A}}$ of A;

(2) to each k-place function symbol F, a function $F_{\mathfrak{A}} : {}^k A \to A$;

(3) to each k-place relation symbol P, a subset $P_{\mathfrak{A}}$ of ${}^k A$.

As a convention, when we denote a model by a Fraktur letter, then we denote the universe of the model by the corresponding italic Roman letter.

In order to define the notions of *satisfaction* and *truth*, let us fix a language $\mathcal{L}$ and a model $\mathfrak{A}$ for $\mathcal{L}$.

The *complexity* of term t is the least n such that $t \in \mathrm{Term}_n^{\mathcal{L}}$. For terms t and for $s \in {}^{<\omega}A$ such that all variables occurring in t belong to $\{v_i \mid i < \ell\mathrm{h}(s)\}$, we define, by recursion on the complexity of t, an element $t_{\mathfrak{A}}^s$ of A:

$$\begin{aligned} c_{\mathfrak{A}}^s &= c_{\mathfrak{A}} \quad \text{for } c \text{ a constant;} \\ v_{i\mathfrak{A}}^{\ s} &= s(i)\,; \\ (F(t_1 \ldots t_n))_{\mathfrak{A}}^s &= F_{\mathfrak{A}}(t_{1\mathfrak{A}}^{\ s}, \ldots t_{n\mathfrak{A}}^{\ s})\,, \end{aligned}$$

where "$F_{\mathfrak{A}}(t_{1\mathfrak{A}}^{\ s}, \ldots t_{n\mathfrak{A}}^{\ s})$" is an abbreviation for "$F_{\mathfrak{A}}(q)$, where $q : n \to A$ and $q(i) = t_{i+1\mathfrak{A}}^{\ s}$ for all $i < n$." Note that $t_{\mathfrak{A}}^s$ is independent of s if no variables occur in t.

Satisfaction. We define, by recursion, for each $n \in \omega$ a relation

$$\mathrm{Sat}_n^{\mathfrak{A}} \subseteq \mathrm{Formula}_n^{\mathcal{L}} \times {}^{<\omega}A.$$

If $\langle \varphi, s\rangle \in \mathrm{Sat}_n^{\mathfrak{A}}$, then the variables having free occurrences in φ must be among $\{v_i \mid i < \ell\mathrm{h}(s)\}$. Also φ must of course belong to $\mathrm{Formula}_n^{\mathcal{L}}$. We shall omit mentioning these two requirements below.

(i) $\langle t_1 = t_2, s\rangle \in \mathrm{Sat}_0^{\mathfrak{A}} \leftrightarrow t_{1\mathfrak{A}}^{\ s} = t_{2\mathfrak{A}}^{\ s}$.

(ii) $\langle P(t_1 \ldots t_k), s\rangle \in \mathrm{Sat}_0^{\mathfrak{A}} \leftrightarrow q \in P_{\mathfrak{A}}$, where $q : k \to A$ and $q(i) = t_{i+1\mathfrak{A}}^{\ s}$ for each $i < k$.

(iii) $\langle \neg\varphi, s\rangle \in \mathrm{Sat}_{n+1}^{\mathfrak{A}} \leftrightarrow \langle \varphi, s\rangle \notin \bigcup\{\mathrm{Sat}_m^{\mathfrak{A}} \mid m \leq n\}$.

(iv) $\langle (\varphi \wedge \psi), s\rangle \in \mathrm{Sat}_{n+1}^{\mathfrak{A}} \leftrightarrow (\langle \varphi, s\rangle \in \bigcup\{\mathrm{Sat}_m^{\mathfrak{A}} \mid m \leq n\} \wedge \langle \psi, s\rangle \in \bigcup\{\mathrm{Sat}_m^{\mathfrak{A}} \mid m \leq n\})$.

(v) $\langle (\exists v_j)\, \varphi, s\rangle \in \mathrm{Sat}_{n+1}^{\mathfrak{A}} \leftrightarrow (\exists s')(s' \supseteq s{\restriction}\mathrm{domain}\,(s)\setminus\{j\} \wedge j \in \mathrm{domain}\,(s') \wedge \langle \varphi, s'\rangle \in \bigcup\{\mathrm{Sat}_m^{\mathfrak{A}} \mid m \leq n\})$.

We let $\mathrm{Sat}^{\mathfrak{A}} = \bigcup\{\mathrm{Sat}_n^{\mathfrak{A}} \mid n \in \omega\}$. We say that $\mathfrak{A}$ *satisfies* $\varphi[s]$ (in symbols, $\mathfrak{A} \models \varphi[s]$) if $\langle \varphi, s\rangle \in \mathrm{Sat}^{\mathfrak{A}}$. If only $v_{i_1}, \ldots, v_{i_n}$ have free occurrences in φ, then we may indicate this by writing $\varphi(v_{i_1}, \ldots, v_{i_n})$ for φ. Moreover we write $\mathfrak{A} \models \varphi[a_1, \ldots, a_n]$ to mean that, for some (or equivalently, every) s such that $s(i_j) = a_j$ for each j, $\mathfrak{A} \models \varphi[s]$.

If a term t has contains no variables, then we write $t_{\mathfrak{A}}$ for $t^s_{\mathfrak{A}}$. If a formula σ has no free occurrences of variables (σ is a *sentence*), then we write $\mathfrak{A} \models \sigma$ for $\mathfrak{A} \models \sigma[s]$. If σ is a sentence and $\mathfrak{A} \models \sigma$ then we say that $\mathfrak{A}$ is a *model of* σ and that σ is *true* in $\mathfrak{A}$.If Σ is a set of sentences then we define

$$\mathfrak{A} \textit{ satisfies } \Sigma \leftrightarrow \mathfrak{A} \models \Sigma \leftrightarrow \mathfrak{A} \text{ is a } \textit{model} \text{ of } \Sigma \leftrightarrow (\forall \sigma \in \Sigma)\, \mathfrak{A} \models \sigma.$$

Exercise 2.3. Theorem 1.4 shows that the definition above of $\mathrm{Sat}^{\mathfrak{A}}$ yields an explicit definition of $\mathrm{Sat}^{\mathfrak{A}}$ from the parameter $\mathfrak{A}$ and so gives us a proper class function $\mathfrak{A} \mapsto \mathrm{Sat}^{\mathfrak{A}}$. Consider the language $\mathcal{L}$ of set theory, which (informally) is the set {"$\in$"}. Think of V as giving a "model" $\mathfrak{V}$ with $|\mathfrak{V}| = V$ and with "$\in$"$_{\mathfrak{V}} = \in$. Can Theorem 1.4 be used define, via clauses like (i)–(v) above, a proper class $\mathrm{Sat}^{\mathfrak{V}} \subseteq \mathrm{Formula}^{\mathcal{L}} \times {}^{<\omega}V$? Explain.

A sentence or a set of sentences of a language $\mathcal{L}$ is *valid* in $\mathcal{L}$ if every model $\mathfrak{A}$ for $\mathcal{L}$ satisfies it. A sentence or a set of sentences of $\mathcal{L}$ is *consistent* (*satisfiable*) in $\mathcal{L}$ if some model $\mathfrak{A}$ for $\mathcal{L}$ satisfies it. It is easy to see by induction that validity and consistency in $\mathcal{L}$ of a sentence σ or set Σ of sentences is independent of $\mathcal{L}$ (for $\mathcal{L}$ containing all symbols in σ or Σ respectively), so we shall usually omit "in $\mathcal{L}$." A sentence σ *logically implies* a sentence τ in $\mathcal{L}$ (in symbols, $\sigma \models_{\mathcal{L}} \tau$) if every model for $\mathcal{L}$ that is a model of σ is a model of τ. Similarly define Σ *logically implies* τ in $\mathcal{L}$ ($\Sigma \models_{\mathcal{L}} \tau$) for sets Σ of sentences and sentences τ. It is easy to see that $\sigma \models_{\mathcal{L}} \tau$ and $\Sigma \models_{\mathcal{L}} \tau$ are independent of $\mathcal{L}$, so we shall usually omit the subscript "$\mathcal{L}$" and the phrase "in $\mathcal{L}$."

A set Σ of sentences has *Henkin witnesses* if whenever $(\exists x)\, \varphi(x) \in \Sigma$ then there is a constant symbol c such that $\varphi(c) \in \Sigma$, where $\varphi(c)$ is the result of substituting c for the free occurrences of x in $\varphi(x)$.

Theorem 2.1 (Henkin Models). (Uses Choice) *Let Σ be a set of sentences of a language $\mathcal{L}$. Suppose that*

(1) every finite subset of Σ is consistent in $\mathcal{L}$;

(2) Σ has Henkin witnesses;

(3) for each sentence σ of $\mathcal{L}$, either $\sigma \in \Sigma$ or $\neg\sigma \in \Sigma$.

Then Σ has a model $\mathfrak{A}$ such that $\mathrm{card}\,(\mathfrak{A}) \leq$ *the cardinal number of the set of constant symbols of $\mathcal{L}$, where we mean by "*$\mathrm{card}\,(\mathfrak{A})$*" not the literal* $\mathrm{card}\,(\mathfrak{A})$ *(namely 2) but rather* $\mathrm{card}\,(A)$.

(The model $\mathfrak{A}$ will be constructed without using Choice. We need Choice to guarantee that the set of all constant symbols of $\mathcal{L}$ has a cardinal number.)

We call a set x *finite* (e.g., in hypothesis (1)), if $\mathrm{card}\,(x) \in \omega$.

Proof. In preparation for the proof of the Completeness Theorem, we shall explicitly record all facts about logical implication needed for the proofs of Theorem 2.1 and Theorem 2.8. (We shall later see that all these facts correspond to facts about a proof-theoretic notion of implication.)

Note that

$$\Delta \text{ consistent } \leftrightarrow \neg(\exists\tau)(\Delta \models \tau \,\wedge\, \Delta \models \neg\tau).$$

For the purpose of listing facts about $\models$, let us take this as the *definition* of consistency.

$$\text{(I)} \qquad \{\tau\} \models \tau$$

$$\text{(II)} \qquad (\Delta_1 \models \tau \,\wedge\, \Delta_1 \subseteq \Delta_2) \,\rightarrow\, \Delta_2 \models \tau$$

Lemma 2.2. *Assume that $\Delta \subseteq \Sigma$ is finite and such that $\Delta \models \tau$. Then $\tau \in \Sigma$.*

Proof. Otherwise hypothesis (3) gives that $\neg\tau \in \Sigma$. By (I) and (II),

$$\Delta \cup \{\neg\tau\} \models \neg\tau \;\wedge\; \Delta \cup \{\neg\tau\} \models \tau.$$

This contradicts hypothesis (1). $\square$

Let us call a formula φ *prime* if φ is either atomic or of the form $(\exists x)\,\psi$. The formulas of $\mathcal{L}$ constitute the smallest set containing the prime formulas of $\mathcal{L}$ and closed under the operations $\varphi \mapsto \neg\varphi$ and $\langle\varphi,\psi\rangle \mapsto (\varphi \wedge \psi)$. This gives rise to a variant notion of complexity of formulas, with respect to which we may use induction and definition by recursion.

A *valuation* for $\mathcal{L}$ is a function v from the set of prime formulas of $\mathcal{L}$ to $\{0,1\}$. Given any valuation v for $\mathcal{L}$ we can define by recursion a canonical $v^* : \mathrm{Formula}^{\mathcal{L}} \rightarrow \{0,1\}$ such that v^* extends v :

$$\begin{aligned} v^*(\varphi) &= v(\varphi) \text{ for } \varphi \text{ prime;} \\ v^*(\neg\varphi) &= 1 - v^*(\varphi)\,; \\ v^*((\varphi \wedge \psi)) &= \min\{v^*(\varphi), v^*(\psi)\}\,. \end{aligned}$$

(For $n \leq m \in \omega,\ m - n$ is the k such that $n + k = m$. It is easy to show the existence and uniquness of such a k.)

A formula φ of $\mathcal{L}$ is *true under* a valuation v if $v^*(\varphi) = 1$. We say that a set Φ of formulas of $\mathcal{L}$ *truth-functionally implies* in $\mathcal{L}$ a formula φ of $\mathcal{L}$ if, for every valuation v for $\mathcal{L}$, if each member of Φ is true under v then φ is true under v. A *tautology* of $\mathcal{L}$ is a formula true under every valuation for $\mathcal{L}$. It is easy to show by induction that truth-functional implication and being a tautology are, in the natural sense, independent of $\mathcal{L}$, so we shall usually omit "in $\mathcal{L}$"and "of $\mathcal{L}$." We write $\Phi \models_{\mathrm{tf}} \varphi$ to mean that Φ truth-functionally implies φ.

Lemma 2.3. *Suppose that Δ is a set of sentences of $\mathcal{L}$ and that τ is a sentence of $\mathcal{L}$. If $\Delta \models_{\mathrm{tf}} \tau$ then $\Delta \models \tau$.*

Proof. Suppose that $\mathfrak{A}$ is a model for $\mathcal{L}$ such that $\mathfrak{A} \models \Delta$ but $\mathfrak{A} \not\models \tau$. Define a valuation v for $\mathcal{L}$ as follows:

$$v(\varphi) = \begin{cases} 0 & \text{if } \varphi \text{ is not a sentence;} \\ 0 & \text{if } \varphi \text{ is a sentence and } \mathfrak{A} \not\models \varphi\,; \\ 1 & \text{if } \varphi \text{ is a sentence and } \mathfrak{A} \models \varphi\,. \end{cases}$$

It is easy to prove by induction on complexity that, for any sentence σ of $\mathcal{L}$, σ is true under v if and only if $\mathfrak{A} \models \sigma$. Hence v witnesses that $\Delta \not\models_{\mathrm{tf}} \tau$. $\square$

Our the next fact in our list is a weakening of Lemma 2.3.

(III) $$(\Delta \text{ finite } \wedge\ \Delta \models_{\mathrm{tf}} \tau) \rightarrow \Delta \models \tau$$

The reason for not taking the full lemma as (III) will be explained later.

Let us write $\models \sigma$ to mean that $\emptyset \models \sigma$, i.e., that σ is valid.

For constants (constant symbols) c_1 and c_2 of $\mathcal{L}$, set

$$c_1 \sim c_2 \ \leftrightarrow\ c_1 = c_2 \in \Sigma\,.$$

Lemma 2.4. *$\sim$ is an equivalence relation.*

Proof. Note that

(IV) $$\models\ c = c \quad \text{for } c \text{ a constant.}$$

By Lemma 2.2, this gives $c \sim c$.

Assume that $c_1 \sim c_2$.

(V) $$\models\ (t_1 = t_2 \rightarrow (\varphi(t_1) \rightarrow \varphi(t_2)) \quad \text{for } \varphi(x) \text{ atomic, } t_1 \text{ and } t_2 \text{ terms without variables}$$

Here $\varphi(t_i)$ is the result of replacing the free occurrences of x in $\varphi(x)$ by occurrences of t_i. Here also we make use of the abbreviation "$\rightarrow$." (See page 2.)

With $x = c_1$ for $\varphi(x)$, we get from (V) that

$$\models\ c_1 = c_2 \rightarrow (c_1 = c_1 \rightarrow c_2 = c_1)\,.$$

Lemma 2.2 then implies that this sentence belongs to Σ. Now one readily checks that $\{\sigma, (\sigma \rightarrow \tau)\} \models_{\mathrm{tf}} \tau$ for any σ and τ. By (III) and two applications of Lemma 2.2, we get that $c_1 = c_2 \in \Sigma$ and so that $c_2 \sim c_1$.

Assume that $c_1 \sim c_2$ and $c_2 \sim c_3$. Applying (V) with $x = c_3$ for $\varphi(x)$, we get that

$$\models\ (c_2 = c_1 \rightarrow (c_2 = c_3 \rightarrow c_1 = c_3))\,.$$

Since $c_2 = c_1 \in \Sigma$ and $c_2 = c_3 \in \Sigma$, it follows by (III) and Lemma 2.2 that $c_1 = c_3 \in \Sigma$ and so that $c_1 \sim c_3$. □

For constants c of $\mathcal{L}$, let $[c] = \{c' \mid c' \sim c\}$. Let

$$A = \{[c] \mid c \text{ is a constant of } \mathcal{L}\}\,.$$

(VI) $$\models\ (\exists v_1)\, v_1 = v_1$$

Lemma 2.5. *The set A is non-empty.*

Proof. By (VI) and Lemma 2.2, the sentence $(\exists v_1)\, v_1 = v_1$ belongs to Σ. Hypothesis (2) yields a constant c of $\mathcal{L}$ such that $c = c \in \Sigma$. Hence there is a constant of $\mathcal{L}$. □

Define $c_{\mathfrak{A}} = [c]$ for each constant c of $\mathcal{L}$.

(VII) $$\begin{array}{l}\models\ (\exists x)\, F(c_1 \ldots c_k) = x \\ \quad \text{for } F \text{ a } k\text{-place function symbol} \\ \quad \text{and } c_1, \ldots, c_k \text{ constants}\end{array}$$

For F and $c_1, \ldots, c_k$ as in (VII), we get by (VII), Lemma 2.2, and hypothesis (2) that there is a constant c with $F(c_1 \ldots c_k) = c \in \Sigma$. Define

$$F_{\mathfrak{A}}([c_1], \ldots, [c_k]) = [c]\,.$$

Here and hereafter we use the following notational convention: $a_1, \ldots, a_k$ denotes the sequence q of length k such that $q(i) = a_{i+1}$ for each $i < k$.

We must show that this does not depend on the representatives $c_1, \ldots, c_k$ and on the choice of c.

(VIII) $\models (t_1 = t_2 \to u(t_1) = u(t_2))$
for $u(x)$ a term, t_1 and t_2 terms without variables

Suppose that $F(c_1 \ldots c_k) = c$ and $F(c'_1 \ldots c'_k) = c'$ both belong to Σ and that $c_i \sim c'_i$ for $1 \leq i \leq k$. For $1 \leq j \leq k+1$, let t_j be the term

$$F(c'_1 \ldots c'_{j-1} c_j \ldots c_k) .$$

(VIII) and (III) give us that $t_j = t_{j+1}$ belongs to Σ for $1 \leq j \leq k$. Let $0 \leq i < k$ and assume that $t_{k+1-i} = t_{k+1} \in \Sigma$. By (V),

$$\models \ (t_{k+1-i} = t_{k+1} \ \to \ (t_{k+1-(i+1)} = t_{k+1-i} \to t_{k+1-(i+1)} = t_{k+1})) .$$

(III) and Lemma 2.2 then give that $t_{k+1-(i+1)} = t_{k+1} \in \Sigma$. By induction we get that $t_1 = t_{k+1} \in \Sigma$, that is, $F(c_1 \ldots c_k) = F(c'_1 \ldots c'_k)$ belongs to Σ. (V) and (III) give that $F(c'_1 \ldots c'_k) = c$ belongs to Σ; (V) and (III) again give that $c = c' \in \Sigma$.

Exercise 2.4. Prove that, for all terms t without variables, $t_{\mathfrak{A}} = [c]$ if and only if $t = c$ belongs to Σ.

We complete the definition of $\mathfrak{A}$ by stipulating that

$$P_{\mathfrak{A}}([c_1], \ldots, [c_k]) \ \leftrightarrow \ P(c_1 \ldots c_k) \in \Sigma .$$

Here we let $P_{\mathfrak{A}}(q) \leftrightarrow q \in P_{\mathfrak{A}}$, and we also use the notational convention introduced above. The proof that the $P_{\mathfrak{A}}$ are well-defined is like the corresponding proof for the $F_{\mathfrak{A}}$.

Lemma 2.6. *Let $\varphi(x)$ be a formula of $\mathcal{L}$, let c be a constant of $\mathcal{L}$, and let $\mathfrak{B}$ be a model for $\mathcal{L}$. Then $\mathfrak{B} \models \varphi[c_{\mathfrak{B}}]$ if and only if $\mathfrak{B} \models \varphi(c)$, where $\varphi(c)$ is the result of replacing the free occurrences of x in $\varphi(x)$ by occurrences of c.*

We omit the proof, an easy induction on the complexity of $\varphi(x)$.

The following lemma completes the proof of the theorem.

Lemma 2.7. *For every sentence σ of $\mathcal{L}$, $\mathfrak{A} \models \sigma$ if and only if $\sigma \in \Sigma$.*

Proof. We proceed by induction of the complexity of σ.

Suppose σ is $t_1 = t_2$. Let $t_{1\mathfrak{A}} = [c_1]$ and $t_{2\mathfrak{A}} = [c_2]$. The $\mathfrak{A} \models \sigma \Leftrightarrow [c_1] = [c_2] \Leftrightarrow c_1 = c_2 \in \Sigma \Leftrightarrow$ (by Exercise 2.4, (V), and (III)) $t_1 = t_2 \in \Sigma$.

The case that σ is $P(t_1 \dots t_k)$ is similar to the case that σ is $t_1 = t_2$.

If σ is $\neg\tau$, then $\mathfrak{A} \models \sigma \Leftrightarrow \mathfrak{A} \not\models \tau \Leftrightarrow \tau \notin \Sigma \Leftrightarrow$ (by (1) and (3)) $\sigma \in \Sigma$.

We have the following truth-functional implications:

$$\{(\tau_1 \wedge \tau_2)\} \models_{\mathrm{tf}} \tau_1 \quad \{(\tau_1 \wedge \tau_2)\} \models_{\mathrm{tf}} \tau_2 \quad \{\tau_1, \tau_2\} \models_{\mathrm{tf}} (\tau_1 \wedge \tau_2)\,.$$

If σ is $(\tau_1 \wedge \tau_2)$ then $\mathfrak{A} \models \sigma \Leftrightarrow (\mathfrak{A} \models \tau_1$ and $\mathfrak{A} \models \tau_2) \Leftrightarrow (\tau_1 \in \Sigma$ and $\tau_2 \in \Sigma)$ $\Leftrightarrow$ (by (III) and Lemma 2.2) $(\tau_1 \wedge \tau_2) \in \Sigma$.

$$\text{(IX)} \qquad \begin{array}{c} \models (\varphi(c) \to (\exists x)\, \varphi(x)) \\ \text{for } c \text{ a constant} \end{array}$$

Suppose that σ is $(\exists x)\, \varphi(x)$. Then $\mathfrak{A} \models \sigma \Leftrightarrow$ there is an $a \in A$ such that $\mathfrak{A} \models \varphi[a] \Leftrightarrow$ there is a constant c of $\mathcal{L}$ such that $\mathfrak{A} \models \varphi[[c]] \Leftrightarrow$ (by Lemma 2.6) there is a constant c of $\mathcal{L}$ such that $\mathfrak{A} \models \varphi(c) \Leftrightarrow$ there is a constant c of $\mathcal{L}$ such that $\varphi(c) \in \Sigma \Leftrightarrow$ ($\Rightarrow$ by (IX), (III), and Lemma 2.2; $\Leftarrow$ by hypothesis (2)) $(\exists x)\, \varphi(x) \in \Sigma$. $\square$

Theorem 2.8. (Uses Choice) *Let $\mathcal{L}$ be a language and let $\mathcal{L}^*$ be obtained from $\mathcal{L}$ by adding* $\max\{\mathrm{card}\,(\mathcal{L}), \aleph_0\}$ *new constant symbols, where* $\mathrm{card}\,(\mathcal{L})$ *is the cardinal number of the set of all non-logical symbols of $\mathcal{L}$. Let Σ be a set of sentences of $\mathcal{L}$ such that every finite subset of Σ is consistent (in $\mathcal{L}$).*

Then there is a set $\Sigma^* \supseteq \Sigma$ of sentences of $\mathcal{L}^*$ such that (1) every finite subset of Σ^* is consistent (in $\mathcal{L}^*$), (2) Σ^* has Henkin witnesses, and (3) for each sentence σ of $\mathcal{L}^*$, either $\sigma \in \Sigma^*$ or $\neg\sigma \in \Sigma^*$.

Proof. Let

$$\kappa = \max\{\aleph_0, \mathrm{card}(\mathcal{L})\}.$$

By Theorem 1.19, $\kappa^{<\omega} = \kappa$. Since κ is the cardinal of the set of all symbols of $\mathcal{L}^*$, the cardinal of the set of all sentences of $\mathcal{L}^*$ is $\leq \kappa^{<\omega}$. There are at least κ sentences of $\mathcal{L}^*$. (Consider sentences $c = c$ for constants c.) Thus κ is the cardinal of the set of all sentences of $\mathcal{L}^*$. Let

$$\alpha \mapsto \sigma_\alpha$$

be a one-one onto function from κ to the set of all sentences of $\mathcal{L}^*$.

Let r be a wellordering of the set of all constant symbols of $\mathcal{L}^*$.

By transfinite recursion, we define sets Σ_α of sentences of $\mathcal{L}^*$ for $\alpha \leq \kappa$. We shall arrange that

(a) $\Sigma_0 = \Sigma$;

(b) $\Sigma_\lambda = \bigcup\{\Sigma_\beta \mid \beta < \lambda\}$ for limit ordinals $\lambda \leq \kappa$;

(c) for $\beta \leq \alpha \leq \kappa$, $\Sigma_\beta \subseteq \Sigma_\alpha$;

(d) for $\alpha \leq \kappa$, every finite subset of Σ_α is consistent (in $\mathcal{L}^*$);

(e) $\operatorname{card}(\Sigma_{\alpha+1} \setminus \Sigma_\alpha) \leq 2$ for $\alpha < \kappa$;

(f) for $\alpha < \kappa$, either $\sigma_\alpha \in \Sigma_{\alpha+1}$ or $\neg\sigma_\alpha \in \Sigma_{\alpha+1}$;

(g) if $\alpha < \kappa$, if σ_α is $(\exists x)\,\varphi(x)$, and if $\sigma_\alpha \in \Sigma_{\alpha+1}$, then $\varphi(c) \in \Sigma_{\alpha+1}$ for some constant c of $\mathcal{L}^*$.

Once we carry out this construction, we can finish the proof by setting $\Sigma^* = \Sigma_\kappa$.

For $\alpha = 0$ and for limit α, we define Σ_α as required by conditions (a) and (b) respectively. Since consistency in $\mathcal{L}$ implies consistency in $\mathcal{L}^*$, (d) holds for $\alpha = 0$. Furthermore (d) holds for limit Σ_λ unless (c) fails for some β and $\alpha < \lambda$ or (d) fails for some $\alpha < \lambda$. for λ in place of κ. This is because, as is not difficult to prove, if Δ is a finite subset of Σ_λ then there is a $\beta < \lambda$ such that $\Delta \subseteq \Sigma_\beta$.

It follows that, however we define Σ_α for successor ordinals α, the smallest ordinal $\gamma \leq \kappa$ such that (a)–(g) fail for the Σ_β, $\beta \leq \gamma$, would have to be a successor ordinal.

Assume then that $\alpha < \kappa$ and that we are given Σ_β, $\beta \leq \alpha$, violating none of (a)–(g).

Suppose first that $\Delta \cup \{\neg\sigma_\alpha\}$ is consistent for every finite $\Delta \subseteq \Sigma_\alpha$. Set

$$\Sigma_{\alpha+1} = \Sigma_\alpha \cup \{\neg\sigma_\alpha\}.$$

Clearly none of (a)–(g) are violated by the Σ_β, $\beta \leq \alpha + 1$.

Before considering the other case, we prove the following lemma.

Lemma 2.9. *Let Δ be a set of sentences and let σ be a sentence. If $\Delta\cup\{\neg\sigma\}$ is inconsistent, then $\Delta \models \sigma$.*

Proof. We use two more facts about $\models$:

(X) $$\Delta \cup \{\sigma\} \models \tau \ \rightarrow\ \Delta \models (\sigma \rightarrow \tau)$$

(XI) $$(\Gamma \models \tau \ \wedge\ (\forall \sigma \in \Gamma)\, \Delta \models \sigma) \ \rightarrow\ \Delta \models \tau$$

We also need that

$$\models_{\mathrm{tf}} ((\neg\sigma \rightarrow \tau) \ \rightarrow\ ((\neg\sigma \rightarrow \neg\tau) \rightarrow \sigma)).$$

Suppose that $\Delta \cup \{\neg\sigma\}$ is inconsistent. For some sentence τ, we have that

$$\Delta \cup \{\neg\sigma\} \models \tau\,;$$
$$\Delta \cup \{\neg\sigma\} \models \neg\tau\,.$$

By (X) we get that $\Delta \models$ both $(\neg\sigma \to \tau)$ and $(\neg\sigma \to \neg\tau)$. By (III) and (XI) we get that $\Delta \models \sigma$. $\square$

Now suppose that there is a finite $\Delta \subseteq \Sigma_\alpha$ such that $\Delta \cup \{\neg\sigma_\alpha\}$ is inconsistent. Fix such a Δ. By Lemma 2.9, we have that $\Delta \models \sigma_\alpha$.

The cardinal number of $\Sigma_\alpha \setminus \Sigma$ is $\leq 2 \cdot \mathrm{card}(\alpha) < \kappa$. Therefore the cardinal number of the set of all new constants of $\mathcal{L}^*$ (i.e., those that are not constants of $\mathcal{L}$) occurring in $\Sigma_\alpha \cup \{\sigma_\alpha\}$ is $< \kappa$. Since κ is the cardinal number of the set of all new constants of $\mathcal{L}^*$, let c_α be the r-least constant of $\mathcal{L}^*$ not occurring in $\Sigma_\alpha \cup \{\sigma_\alpha\}$.

Let

$$\Sigma_{\alpha+1} = \Sigma_\alpha \cup \{\sigma_\alpha\},$$

unless σ_α is $(\exists x)\,\varphi_\alpha(x)$ for some formula φ_α, in which case let

$$\Sigma_{\alpha+1} = \Sigma_\alpha \cup \{\sigma_\alpha, \varphi_\alpha(c_\alpha)\}.$$

If we can prove that every finite subset of $\Sigma_{\alpha+1}$ is consistent, then we will have shown that (a)–(g) do not fail for the Σ_β, $\beta \leq \alpha+1$, and so we will have completed the proof of the theorem.

Assume that $\Delta' \cup \{\sigma_\alpha\}$ is inconsistent for some finite subset Δ' of Σ_α. By (XI), (III), and the fact that $\{\neg\neg\sigma_\alpha\} \models_{\mathrm{tf}} \sigma_\alpha$, we get that $\Delta' \cup \{\neg\neg\sigma_\alpha\}$ is inconsistent. By Lemma 2.9, we get that $\Delta' \models \neg\sigma_\alpha$. But then $\Delta \cup \Delta'$ is an inconsistent finite subset of Σ_α.

(XII) $\quad \Delta \cup \{\psi(c)\} \models \tau \;\to\; \Delta \cup \{(\exists x)\,\psi(x)\} \models \tau$
for c is a constant not occurring in Δ, $\psi(x)$, or τ

(If $\mathfrak{B}$ is a model satisfying $\Delta \cup \{(\exists x)\psi(x)\}$ but not τ, then let $b \in B$ be such that $\mathfrak{B} \models \psi[b]$. Let $\mathfrak{B}'$ be like $\mathfrak{B}$, except that $c_{\mathfrak{B}'} = b$. Then $\mathfrak{B}'$ satisfies $\Delta \cup \{\psi(c)\}$ but not τ.)

Assume that some finite subset of $\Sigma_{\alpha+1}$ is inconsistent. Then $\Sigma_{\alpha+1} = \Sigma_\alpha \cup \{\sigma_\alpha, \varphi_\alpha(c_\alpha)\}$, and there is a finite $\bar{\Delta} \subseteq \Sigma_\alpha$ and there is a sentence τ such that

$$\bar{\Delta} \cup \{\sigma_\alpha, \varphi_\alpha(c_\alpha)\} \models \tau\,;$$
$$\bar{\Delta} \cup \{\sigma_\alpha, \varphi_\alpha(c_\alpha)\} \models \neg\tau\,.$$

Using the the truth-functional implication $\{\tau, \neg\tau\} \models_{\mathrm{tf}} \tau'$, we may assume that c_α does not occur in τ. By (XII) we have

$$\bar{\Delta} \cup \{\sigma_\alpha, (\exists x)\, \varphi_\alpha(x)\} \models \tau\,;$$
$$\bar{\Delta} \cup \{\sigma_\alpha, (\exists x)\, \varphi_\alpha(x)\} \models \neg\tau\,.$$

But σ_α is $(\exists x)\, \varphi_\alpha(x)$, so we have the contradiction that $\Delta_\alpha \cup \{\sigma_\alpha\}$ is inconsistent. □

Theorem 2.10. (Compactness I and Weak Löwenheim–Skolem Theorem) (Uses Choice) *Let Σ be a set of sentences of a language $\mathcal{L}$ such that every finite subset of Σ is consistent. Then there is a model $\mathfrak{A}$ of Σ such that* $\mathrm{card}(\mathfrak{A}) \leq \max\{\aleph_0, \mathrm{card}(\mathcal{L})\}$.

Proof. Let $\mathcal{L}^*$ be as in the statement of Theorem 2.8. Let Σ^* be given by that theorem. Let $\mathfrak{A}^*$ be the model of Σ^* given by Theorem 2.1. Let $\mathfrak{A}$ be the *reduct* of $\mathfrak{A}^*$ to $\mathcal{L}$. Clearly $\mathfrak{A} \models \Sigma$. □

Theorem 2.11 (Compactness II). (Uses Choice) *Let Σ be a set of sentences and let σ be a sentence. If $\Sigma \models \sigma$ then there is a finite $\Delta \subseteq \Sigma$ such that $\Delta \models \sigma$.*

Proof. Suppose that $\Sigma \models \sigma$. Then $\Sigma \cup \{\neg\sigma\}$ is inconsistent. By Theorem 2.10, there is a finite $\Delta \subseteq \Sigma$ such that $\Delta \cup \{\neg\sigma\}$ is inconsistent. But then $\Delta \models \sigma$. □

Exercise 2.5. Let $\mathcal{L}$ be any language. A class K of models for $\mathcal{L}$ is EC (is an *elementary class*) if there is a sentence σ of $\mathcal{L}$ such that

$$K = \{\mathfrak{A} \mid \mathfrak{A} \models \sigma\}\,.$$

A class K is EC_Δ if there is a set Σ of sentences of $\mathcal{L}$ such that

$$K = \{\mathfrak{A} \mid \mathfrak{A} \models \Sigma\}\,.$$

Which of the following are EC_Δ?

(i) $\{\mathfrak{A} \mid A \text{ is infinite}\}$;

(ii) $\{\mathfrak{A} \mid A \text{ is finite}\}$.

Show that neither is EC.

Theorem 2.12. *Assume that ZFC (i.e., the set of axioms of ZFC) is consistent. For variables x, let* $\text{Number}\,(x)$ *be the formula "x is a natural number."*

There is a model $\mathfrak{A}$ of ZFC and an $a \in A$ such that $\mathfrak{A} \models \text{Number}\,[a]$ and such that $\in_{\mathfrak{A}} \restriction \{b \mid b \in_{\mathfrak{A}} a\}$ is not wellfounded.

Proof. For $n \in \omega$, let $\chi_n(x)$ be the formula "$x = n$." ($\chi_n(x)$ is defined by recursion on n.) Let $\mathcal{L}^*$ be the result of adding to the language of set theory a constant c. Let

$$\Sigma = \text{ZFC} \cup \{\text{Number}\,(c)\} \cup \{(\forall v_0)(\chi_n(v_0) \rightarrow v_0 \in c) \mid n \in \omega\}\,.$$

Let Δ be a finite subset of Σ. Then there is some $m \in \omega$ such that

$$\Delta \subseteq \text{ZFC} \cup \{\text{Number}\,(c)\} \cup \{(\forall v_0)(\chi_n(v_0) \rightarrow v_0 \in c) \mid n < m\}\,.$$

Let $\mathfrak{B}$ be a model of ZFC. For each $n \in \omega$ there is a unique $b \in B$ such that $\mathfrak{B} \models \chi_n[b]$; let $n^{\mathfrak{B}}$ be this unique b. Expand $\mathfrak{B}$ to a model $\mathfrak{B}^*$ for $\mathcal{L}^*$ by letting $c_{\mathfrak{B}^*} = m^{\mathfrak{B}}$. Clearly $\mathfrak{B}^* \models \Delta$.

Since every finite subset of Σ is consistent, there is by Theorem 2.10 a model $\mathfrak{A}^*$ of Σ. Let $\mathfrak{A}$ be the reduct of $\mathfrak{A}^*$ to $\mathcal{L}$, and let $a = c_{\mathfrak{A}^*}$.

To see that $\in_{\mathfrak{A}} \restriction \{b \mid b \in_{\mathfrak{A}} a\}$ is not wellfounded, let

$$y = \{b \mid b \in_{\mathfrak{A}} a \;\wedge\; (\forall n \in \omega)\mathfrak{A} \not\models \chi_n[b]\}.$$

Since the $\in_{\mathfrak{A}}$-immediate predecessor of a belongs to y, y is nonempty. For any $b \in y$, the $\in_{\mathfrak{A}}$-immediate predecessor of b belongs to y, so y has no $\in_{\mathfrak{A}}$-least element. $\square$

Remark. If $\mathfrak{A}$ and a are as in the statement of Theorem 2.12, then a is a *non-standard* natural number of $\mathfrak{A}$. In §3, we shall construct models with non-standard real numbers.

If $\mathfrak{A}$ and $\mathfrak{B}$ are models for a language $\mathcal{L}$, then $\mathfrak{A}$ and $\mathfrak{B}$ are *elementarily equivalent* ($\mathfrak{A} \equiv \mathfrak{B}$) if they satisfy the same sentences of $\mathcal{L}$.

Theorem 2.13. *Let $\mathcal{L}$ be a language and let $\kappa = \max\{\aleph_0, \text{card}(\mathcal{L})\}$. Every model for $\mathcal{L}$ is elementarily equivalent to a model of cardinal $\leq \kappa$.*

Proof. Let $\mathfrak{B}$ be a model for $\mathcal{L}$. The *theory of* $\mathfrak{B}$ ($\text{Th}(\mathfrak{B})$), the set of all sentences σ such that $\mathfrak{B} \models \sigma$, is consistent. Apply Theorem 2.10. $\square$

Formal Deduction

Fix a language $\mathcal{L}$.

Logical Axioms:

(1) All tautologies.

(2) Identity Axioms:

  (a) $t = t$
    for t a term;

  (b) $(t_1 = t_2 \to (\varphi(t_1, y_1, \ldots, y_n) \to \varphi(t_2, y_1, \ldots, y_n)))$
    for t_1 and t_2 terms and $\varphi(x, y_1, \ldots, y_n)$ an atomic formula.

(3) Quantifier Axioms:

$$(\psi(t, y_1, \ldots, y_n) \to (\exists x)\, \psi(x, y_1, \ldots, y_n)),$$

for $\psi(x, y_1, \ldots, y_n)$ a formula and t a term such that no occurrence of a variable in t gives a bound occurrence of the variable in $\psi(t, y_1, \ldots, y_n)$.

Rules:

(1) Modus Ponens: $\dfrac{\varphi \quad (\varphi \to \psi)}{\psi}$

for φ and ψ formulas;

(2) Quantifier Rule: $\dfrac{(\varphi \to \psi)}{((\exists x)\varphi \to \psi)}$

for φ and ψ formulas with x not free in ψ.

Remark. In stating the axioms and rules, we have used abbreviations involving the symbol "$\to$" (introduced on page 2).

A *deduction* in $\mathcal{L}$ from a set Σ of sentences is a finite sequence of formulas (the *lines* of the deduction) such that every formula in the sequence either (i) belongs to Σ, (ii) is a logical axiom, or (iii) follows from earlier formulas by one of the two rules. A *deduction* in $\mathcal{L}$ *of* a sentence τ from Σ is a deduction in $\mathcal{L}$ from Σ with last line τ.

A set Σ of sentences *deductively implies* in $\mathcal{L}$ a sentence τ $(\Sigma \vdash_{\mathcal{L}} \tau)$ if there is a deduction in $\mathcal{L}$ of τ from Σ.

Remark. It will turn out that deductive implication is independent of $\mathcal{L}$, but this is not as easy to prove as the corresponding fact for the semantical notion of logical implication.

Theorem 2.14 (Soundness). *For any language $\mathcal{L}$, if $\Sigma \vdash_{\mathcal{L}} \tau$ then $\Sigma \models \tau$.*

Proof. Let $\mathcal{D}$ be a deduction from Σ in $\mathcal{L}$ and let $\mathfrak{A}$ be any model of Σ. By induction one can show that, for all lines φ of $\mathcal{D}$ and for every s (with large enough domain), $\mathfrak{A} \models \varphi[s]$. This is trivial for $\varphi \in \Sigma$ and is easily checked for logical axioms. Moreover it is easy to see that applications of the rules preserve this property. $\square$

Theorem 2.15. *For any language $\mathcal{L}$,* (I)–(XII) *hold with "$\vdash_{\mathcal{L}}$" in place of "$\models$."*

Remark. The modified (III), like the original (III), remains true if the restriction that Δ be finite, is removed. This is because—as is not difficult to show—compactness holds for truth-functional implication. Our reason for the restriction to finite Δ is to save ourselves the effort of proving the unrestricted version.

Proof. (I), (II), and (XI) follow directly from the notion of a deduction, and do not depend on our particular axioms and rules.

(IV) and (V) are Identity Axioms, and (VIII) follows from Identity Axioms (a) and (b) using Modus Ponens.

For (III), suppose that $\Delta \models_{\mathrm{tf}} \tau$ with Δ finite. Let Δ be $\{\sigma_i \mid i < n\}$. Then

$$(\sigma_0 \to (\sigma_1 \to \ldots \to (\sigma_{n-1} \to \tau) \cdots)$$

is a tautology. By n applications of Modus Ponens, we can get a deduction of τ from Δ.

(VI) follows by Modus Ponens from the Identity Axiom $v_1 = v_1$ and the Quantifier Axiom $(v_1 = v_1 \to (\exists v_1)\, v_1 = v_1)$.

For (VII), note that

$$F(c_1, \ldots, c_k) = F(c_1, \ldots, c_k)$$

is an Identity Axiom and that

$$(F(c_1, \ldots, c_k) = F(c_1, \ldots, c_k) \ \to\ (\exists x)\, F(c_1, \ldots, c_k) = x)$$

is a Quantifier Axiom. (VI) follows from these axioms by Modus Ponens.

(IX) is a Quantifier Axiom.

(X) is commonly called the *Deduction Theorem.* To prove it, let $\mathcal{D}$ be a deduction in $\mathcal{L}$ of τ from $\Delta \cup \{\sigma\}$. Get a new sequence $\mathcal{D}'$ of formulas by replacing each line φ of $\mathcal{D}$ by $(\sigma \to \varphi)$. We shall show how to turn $\mathcal{D}'$ into a deduction of $(\sigma \to \tau)$ from Δ by inserting additional lines.

If a line φ of $\mathcal{D}$ belongs to Δ or is a logical axiom, then insert φ and the tautology $(\varphi \to (\sigma \to \varphi))$. The line $(\sigma \to \varphi)$ then comes by Modus Ponens.

If a line of $\mathcal{D}$ is σ, then the corresponding line of $\mathcal{D}'$ is the tautology $(\sigma \to \sigma)$.

If a line φ of $\mathcal{D}$ comes from earlier lines ψ and $(\psi \to \varphi)$ by Modus Ponens, then insert the tautology

$$((\sigma \to \psi) \to ((\sigma \to (\psi \to \varphi)) \to (\sigma \to \varphi))) \tag{$\dagger$}$$

and the formula

$$((\sigma \to (\psi \to \varphi)) \to (\sigma \to \varphi))\,. \tag{$\ddagger$}$$

($\ddagger$) comes from the ($\dagger$) and $(\sigma \to \psi)$ by Modus Ponens, and $(\sigma \to \varphi)$ then comes from the ($\ddagger$) and $(\sigma \to (\psi \to \varphi))$ by another application of Modus Ponens.

Suppose finally that a line of $\mathcal{D}$ is $((\exists x)\varphi \to \psi)$ and that it comes from an earlier line $(\varphi \to \psi)$ by the Quantifier Rule. That earlier line corresponds to the line $(\sigma \to (\varphi \to \psi))$ of $\mathcal{D}'$. Insert the following lines:

$$\begin{array}{l}
((\sigma \to (\varphi \to \psi)) \to (\varphi \to (\sigma \to \psi))) \\
(\varphi \to (\sigma \to \psi)) \\
((\exists x)\varphi \to (\sigma \to \psi)) \\
(((\exists x)\varphi \to (\sigma \to \psi)) \to (\sigma \to ((\exists x)\varphi \to \psi))) \\
(\sigma \to ((\exists x)\varphi \to \psi)))
\end{array}$$

The first and fourth of these lines are tautologies. The second and fifth come by Modus Ponens. The third comes by the Quantifier Rule. Finally, the line $(\sigma \to \varphi)$ comes by Modus Ponens.

It remains only to show that (XII) holds. Assume that $\Sigma \cup \{\psi(c)\} \vdash_{\mathcal{L}} \tau$ and that the conditions of (XII) are met. By (X) we have that $\Sigma \vdash_{\mathcal{L}} (\psi(c) \to \tau)$. Let $\mathcal{D}$ be a deduction witnessing this fact. Let y be a variable not occurring in $\mathcal{D}$. We get a deduction $\mathcal{D}'$ from Σ with last line $(\psi(y) \to \tau)$ by replacing each occurrence of c in $\mathcal{D}$ by an occurrence of y. Applying the Quantifier Rule to the last line of $\mathcal{D}'$, we get $((\exists y)\psi(y) \to \tau)$. From this, the Quantifier Axiom $(\psi(x) \to (\exists y)\psi(y))$, and tautologies and Modus Ponens, we get $(\psi(x) \to \tau)$. The Quantifier Rule now gives $((\exists x)\psi(x) \to \tau)$. $\square$

Let us say that a set Σ of sentences of a language $\mathcal{L}$ is *deductively consistent* in $\mathcal{L}$ if there is no sentence τ of $\mathcal{L}$ such that $\Sigma \vdash_{\mathcal{L}} \tau$ and $\Sigma \vdash_{\mathcal{L}} \neg\tau$. Otherwise Σ is *deductively inconsistent* in $\mathcal{L}$. Since deductions are finite, a Σ is deductively consistent in $\mathcal{L}$ if and only if every finite subset of Σ is deductively consistent in $\mathcal{L}$.

Theorem 2.16. (Uses Choice) *Let Σ be a set of sentences of a language $\mathcal{L}$. Suppose that*

(1) Σ is deductively consistent in $\mathcal{L}$;

(2) Σ has Henkin witnesses;

(3) for each sentence σ of $\mathcal{L}$, either $\sigma \in \Sigma$ or $\neg\sigma \in \Sigma$.

Then Σ has a model $\mathfrak{A}$ such that $\operatorname{card}(\mathfrak{A}) \leq$ *the cardinal number of the set of constant symbols of $\mathcal{L}$.*

(As with Theorem 2.1, Choice is needed only to guarantee that the set of all constant symbols of $\mathcal{L}$ has a cardinal number.)

Proof. The proof is exactly like that of Theorem 2.1, using Theorem 2.15. $\square$

Theorem 2.17. *Let Σ be a set of sentences of a language $\mathcal{L}$ such that Σ is deductively consistent in $\mathcal{L}$. Let $\mathcal{L}^*$ be obtained from $\mathcal{L}$ by adding new constant symbols. Then Σ is deductively consistent in $\mathcal{L}^*$.*

Proof. Assume that Σ is deductively inconsistent in $\mathcal{L}^*$ Then there is a sentence τ, which we may without loss of generality assume to be a sentence of $\mathcal{L}$, such that $\Sigma \vdash_{\mathcal{L}*} \tau$ and $\Sigma \vdash_{\mathcal{L}*} \neg\tau$. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be deductions witnessing these facts. Let $c_1, \ldots, c_n$ be distinct and be all the constants of $\mathcal{L}^*$ occurring in either of $\mathcal{D}_1$ or $\mathcal{D}_2$ that are not constants of $\mathcal{L}$. Let $y_1, \ldots, y_n$ be distinct variables not occurring in $\mathcal{D}_1$ or $\mathcal{D}_2$. Obtain $\mathcal{D}_1'$ and $\mathcal{D}_2'$ from $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively by replacing, for each i, each occurrence of c_i by an occurrence of y_i. Then $\mathcal{D}_1'$ and $\mathcal{D}_2'$ witness that $\Sigma \vdash_{\mathcal{L}} \tau$ and $\Sigma \vdash_{\mathcal{L}} \neg\tau$ respectively.

Theorem 2.18. (Uses Choice) *Let $\mathcal{L}$ be a language and let $\mathcal{L}^*$ be obtained from $\mathcal{L}$ by adding* $\max\{\operatorname{card}(\mathcal{L}), \aleph_0\}$ *new constant symbols. Let Σ be a set of sentences of $\mathcal{L}$ such that Σ is deductively consistent in $\mathcal{L}$.*

Then there is a set $\Sigma^* \supseteq \Sigma$ of sentences of $\mathcal{L}^*$ such that (1) Σ^* is deductively consistent in $\mathcal{L}^*$, (2) Σ^* has Henkin witnesses, and (3) for each sentence σ of $\mathcal{L}^*$, either $\sigma \in \Sigma^*$ or $\neg\sigma \in \Sigma^*$.

Proof. The proof is exactly like that of Theorem 2.8, using Theorem 2.15 and using Theorem 2.17 to get that $\Sigma_0 = \Sigma$ is deductively consistent in $\mathcal{L}^*$. $\square$

Theorem 2.19. (Uses Choice) *Let* Σ *be a set of sentences of a language* $\mathcal{L}$ *such* Σ *is deductively consistent in* $\mathcal{L}$*. Then there is a model* $\mathfrak{A}$ *of* Σ *such that* $\mathrm{card}(\mathfrak{A}) \leq \max\{\aleph_0, \mathrm{card}(\mathcal{L})\}$.

Proof. The proof is like that of Theorem 2.10. $\square$

Theorem 2.20 (Gödel Completeness Theorem). (Uses Choice.) *Let* Σ *be a set of sentences of a language* $\mathcal{L}$ *and let* σ *be a sentence of* $\mathcal{L}$*. If* $\Sigma \models \sigma$ *then* $\Sigma \vdash_{\mathcal{L}} \sigma$.

Proof. Assume that $\Sigma \nvdash_{\mathcal{L}} \sigma$. Then, by the analogue of Lemma 2.9, $\Sigma \cup \{\neg\sigma\}$ is deductively consistent in $\mathcal{L}$. By Theorem 2.19, there is a model $\mathfrak{A}$ for $\mathcal{L}$ such that $\mathfrak{A} \models \Sigma \cup \{\neg\sigma\}$. But then $\Sigma \not\models \sigma$. $\square$

Because of the Soundness and Completenenss Theorems, the symbol "$\vdash_{\mathcal{L}}$," is superfluous, and we shall make no further use of it.

Exercise 2.6. Let $\mathcal{L}$ be a language with a one-place relation symbol F. Give a deduction witnessing the following

$$\{\neg(\exists v_1)\neg F(v_1)\} \ \vdash_{\mathcal{L}} \ \neg(\exists v_2)\neg F(v_2).$$

Exercise 2.7. Suppose we replaced our Quantifier Rule with the following additional Logical Axioms:

$$((\varphi \to \psi) \to ((\exists x)\varphi \to \psi))$$
$$\text{for } x \text{ not occurring free in } \psi.$$

Would Soundness still hold? Would Completeness still hold? Prove your answers.

3 Model theory

For the next several definitions, fix a language $\mathcal{L}$.

If $\mathfrak{A}$ and $\mathfrak{B}$ are models for $\mathcal{L}$, then $\mathfrak{A}$ and $\mathfrak{B}$ are *isomorphic* ($\mathfrak{A} \cong \mathfrak{B}$) if there is a one-one onto $f : A \to B$ such that

(1) $f(c_\mathfrak{A}) = c_\mathfrak{B}$ for every constant c of $\mathcal{L}$;

(2) $f(F_\mathfrak{A}(a_1, \ldots, a_k)) = F_\mathfrak{B}(f(a_1), \ldots, f(a_k))$ for all $k \in \omega \setminus \{0\}$, all k-place function symbols of F of $\mathcal{L}$, and all $(a_1, \ldots, a_k) \in {}^kA$;

(3) $P_\mathfrak{A}(a_1, \ldots, a_k)) \leftrightarrow P_\mathfrak{B}(f(a_1), \ldots, f(a_k))$ for all $k \in \omega \setminus \{0\}$, all k-place relation symbols P of $\mathcal{L}$, and all $(a_1, \ldots, a_k) \in {}^kA$.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be models for $\mathcal{L}$. The model $\mathfrak{A}$ is a *submodel* of $\mathfrak{B}$ ($\mathfrak{A} \subseteq \mathfrak{B}$) if $A \subseteq B$ and

(1) $c_\mathfrak{A} = c_\mathfrak{B}$ for all constants c of $\mathcal{L}$;

(2) $F_\mathfrak{A} = F_\mathfrak{B} \restriction {}^kA$ for all $k \in \omega \setminus \{0\}$ and all k-place function symbols F of $\mathcal{L}$;

(3) $P_\mathfrak{A} = P_\mathfrak{B} \cap {}^kA$ for all $k \in \omega \setminus \{0\}$ and all k-place relation symbols P of $\mathcal{L}$.

We say that $\mathfrak{A}$ is an *elementary submodel* of $\mathfrak{B}$ ($\mathfrak{A} \prec \mathfrak{B}$) if $\mathfrak{A} \subseteq \mathfrak{B}$ and, for every formula $\varphi(x_1, \ldots, x_n)$ of $\mathcal{L}$ and any elements $a_1, \ldots, a_n$ of A,

$$\mathfrak{A} \models \varphi[a_1, \ldots, a_n] \leftrightarrow \mathfrak{B} \models \varphi[a_1, \ldots, a_n].$$

The condition that $\mathfrak{A} \subseteq \mathfrak{B}$ can be weakend to $A \subseteq B$ without affecting the defined concept. To see why this is so, note, for example, that (2) in the definition of $\mathfrak{A} \subseteq \mathfrak{B}$ can be deduced using the formula $F(v_1 \ldots v_n) = v_{n+1}$.

If $\mathfrak{A}$ is a model for $\mathcal{L}$, let $\mathcal{L}_A$ be the language resulting from adding to $\mathcal{L}$ distinct new constants c^a for each $a \in A$. (This can be done in a definable fashion.) The *elementary diagram* of $\mathfrak{A}$ is $\mathrm{Th}(\mathfrak{A}_A)$, where $\mathfrak{A}_A$ is the *expansion* of $\mathfrak{A}$ resulting from setting $c^a_{\mathfrak{A}_A} = a$.

Theorem 3.1. *Let $\mathfrak{A}$ be a model for a language $\mathcal{L}$. Suppose that $\mathfrak{B}^*$ is a model for $\mathcal{L}_A$ such that $\mathfrak{B}^*$ is a model of the elementary diagram of $\mathfrak{A}$. Let $\mathfrak{B}$ be the reduct of $\mathfrak{B}^*$ to $\mathcal{L}$. Then there is a $\mathfrak{B}' \cong \mathfrak{B}$ such that $\mathfrak{A} \prec \mathfrak{B}'$.*

Proof. We may assume without loss of generality that $A \cap B = \emptyset$.

Define $f : A \to B$ by setting

$$f(a) = c^a_{\mathfrak{B}^*},$$

for $a \in A$. For any formula $\varphi(x_1, \ldots, x_n)$ of $\mathcal{L}$ and for any elements $a_1, \ldots, a_n$ of A,

$$\begin{aligned}
\mathfrak{A} \models \varphi[a_1, \ldots, a_n] &\leftrightarrow \mathfrak{A}_A \models \varphi(c^{a_1}, \ldots, c^{a_n}) \\
&\leftrightarrow \mathfrak{B}^* \models \varphi(c^{a_1}, \ldots, c^{a_n}) \\
&\leftrightarrow \mathfrak{B}^* \models \varphi[c^{a_1}_{\mathfrak{B}^*}, \ldots, c^{a_n}_{\mathfrak{B}^*}] \\
&\leftrightarrow \mathfrak{B} \models \varphi[c^{a_1}_{\mathfrak{B}^*}, \ldots, c^{a_n}_{\mathfrak{B}^*}] \\
&\leftrightarrow \mathfrak{B} \models \varphi[f(a_1), \ldots, f(a_n)]\,.
\end{aligned}$$

Taking for φ the formula $v_1 = v_2$, we get that f is one-one.

Let $C = \operatorname{range}(f)$. Let $\mathfrak{C}$ be the model with universe C such that $f : \mathfrak{A} \cong \mathfrak{C}$. To see that $\mathfrak{C} \prec \mathfrak{B}$, let φ be a formula of $\mathcal{L}$ and let $b_1, \ldots, b_n$ be elements of C. Then

$$\mathfrak{C} \models \varphi[b_1, \ldots, b_n] \leftrightarrow \mathfrak{A} \models \varphi[f^{-1}(b_1), \ldots, f^{-1}(b_n)] \leftrightarrow \mathfrak{B} \models [b_1, \ldots, b_n]\,.$$

Let $B' = (B \setminus C) \cup A$. Define $\mathfrak{B}'$ as follows. Let $c_{\mathfrak{B}'} = c_{\mathfrak{A}}$ for each constant c of $\mathcal{L}$. For $b' \in B'$ let

$$g(b') = \begin{cases} b' & \text{if } b' \in B\,; \\ f(b') & \text{if } b' \in A\,. \end{cases}$$

Now define the interpretation of function and relation symbols by setting

$$\begin{aligned}
F_{\mathfrak{B}'}(b'_1, \ldots, b'_k) &= g^{-1}(F_{\mathfrak{B}}(g(b'_1), \ldots, g(b'_k)))\,; \\
P_{\mathfrak{B}'}(b'_1, \ldots, b'_k) &\leftrightarrow P_{\mathfrak{B}}(g(b'_1), \ldots, g(b'_k))\,.
\end{aligned}$$

It is easy to see that $\mathfrak{B}'$ is as required. $\square$

Theorem 3.2. (Uses Choice) *Let*

$$\mathcal{L} = \{\mathbf{0}, \mathbf{1}, <, +, \cdot\}.$$

Let $\mathfrak{R}$ be the obvious model for $\mathcal{L}$ whose universe is the set $\mathbb{R}$ of all real numbers. The model $\mathfrak{R}$ has a non-archimedean elementary extension; i.e., there is a $\mathfrak{A}$ such that $\mathfrak{R} \prec \mathfrak{A}$ and

$$(\exists a \in A)(\forall n \in \omega)(\mathbf{0}_{\mathfrak{A}} <_{\mathfrak{A}} a \wedge \underbrace{a +_{\mathfrak{A}} \cdots +_{\mathfrak{A}} a}_{n} <_{\mathfrak{A}} \mathbf{1}_{\mathfrak{A}})\,.$$

Proof. Let $\mathcal{L}^*$ be the result of adding to $\mathcal{L}_{\mathbb{R}}$ a new constant c. Let

$$\Sigma = \mathrm{Th}(\mathfrak{R}_{\mathbb{R}}) \cup \{\mathbf{0}{<}c\} \cup \{\underbrace{c+\cdots+c}_{n}{<}\mathbf{1} \mid n \in \omega\}\,.$$

Every finite subset of Σ is satisfied by some expansion of $\mathfrak{R}_{\mathbb{R}}$. By Theorem 2.10, let $\mathfrak{B}^* \models \Sigma$. Apply Theorem 3.1. $\square$

Theorem 3.3. (Upward Löwenheim–Skolem–Tarski Theorem) *Let $\mathfrak{A}$ be a model for a language $\mathcal{L}$ and suppose that κ is a cardinal number such that*

$$\kappa \geq \mathrm{card}(\mathfrak{A}) \geq \aleph_0 \,\wedge\, \kappa \geq \mathrm{card}(\mathcal{L})\,.$$

Then there is an elementary extension $\mathfrak{B}$ of $\mathfrak{A}$ such that $\mathrm{card}(\mathfrak{B}) = \kappa$.

Proof. Let $\mathcal{L}^*$ be the result of adjoining to $\mathcal{L}_A$ distinct new constants c_α, $\alpha < \kappa$. Let

$$\Sigma = \mathrm{Th}(\mathfrak{A}_A) \cup \{c_\alpha \neq c_\beta \mid \alpha \neq \beta\}\,.$$

Every finite subset of Σ is satisfied by an expansion of $\mathfrak{A}_A$. By Theorem 2.10, this means that there is a model $\mathfrak{C}$ for Σ with $\mathrm{card}(\mathfrak{C}) \leq \kappa$. But then $\mathrm{card}(\mathfrak{C}) = \kappa$. Apply Theorem 3.1 to get $\mathfrak{B}$. $\square$

If $\mathfrak{A}$ is a model for a language $\mathcal{L}$ and $\emptyset \neq B \subseteq A$, then a necessary and sufficient condition for B to be the universe of a submodel of $\mathfrak{A}$ is that (i) $c_{\mathfrak{A}}$ belongs to B for each constant c of $\mathcal{L}$ and (ii) that B is closed under $F_{\mathfrak{A}}$ for each function symbol F of $\mathfrak{A}$. The following theorem gives a necessary and sufficient condition for B to be the universe of an elementary submodel of $\mathfrak{A}$.

Theorem 3.4. *Let $\mathfrak{A}$ be a model for a language $\mathcal{L}$. Let B be a non-empty subset of A. Then the following are equivalent:*

(1) There is a (unique) $\mathfrak{B} \prec \mathfrak{A}$ such that $|\mathfrak{B}| = B$.

(2) For every formula $\varphi(y, x_1, \ldots, x_n)$ of $\mathcal{L}$ and any elements $b_1, \ldots, b_n$ of B, there is an element b of B such that

$$\mathfrak{A} \models ((\exists y)\varphi)[b_1, \ldots, b_n] \;\rightarrow\; \mathfrak{A} \models \varphi[b, b_1, \ldots, b_n]\,.$$

Proof. That (1) implies (2) is easy to see. Suppose then that (2) holds.

We first argue that (i) and (ii) of the paragraph preceding the theorem are satisfied, and so there is a submodel $\mathfrak{B}$ of $\mathfrak{A}$ with universe B. If c

is a constant then $\mathfrak{A} \models (\exists x)\, x = c$. By (2) there is a $b \in B$ such that $\mathfrak{A} \models (x = c)[b]$. Hence $c_{\mathfrak{A}} \in B$. If F is a k-place function symbol and $b_1, \ldots, b_k$ are elements of B, then $\mathfrak{A} \models ((\exists y) F(x_1, \ldots, x_k) = y)[b_1, \ldots, b_k]$. By (2) there is a $b \in B$ such that $\mathfrak{A} \models (F(x_1, \ldots, x_k) = y)[b, b_1, \ldots, b_k]$. Thus $b = F_{\mathfrak{A}}(b_1, \ldots, b_k)$. This argument shows that B is closed under $F_{\mathfrak{A}}$.

By induction on complexity, we show that for all formulas $\varphi(x_1, \ldots, x_n)$ of $\mathcal{L}$ and any elements $b_1, \ldots, b_n$ of B,

$$\mathfrak{B} \models \varphi[b_1, \ldots, b_n] \leftrightarrow \mathfrak{A} \models \varphi[b_1, \ldots, b_n]\,.$$

For φ atomic this follows from $\mathfrak{B} \subseteq \mathfrak{A}$. The cases that φ is $\neg\psi$ and that φ is $(\psi \wedge \chi)$ are straightforward. Suppose that φ is $(\exists y)\, \psi(y, x_1, \ldots, x_n)$. If $\mathfrak{B} \models \varphi[b_1, \ldots, b_n]$, then it follows easily that $\mathfrak{A} \models \varphi[b_1, \ldots, b_n]$. Assume that $\mathfrak{A} \models \varphi[b_1, \ldots, b_n]$. By (2) there is an $b \in B$ such that $\mathfrak{A} \models \psi[b, b_1, \ldots, b_n]$. By induction, $\mathfrak{B} \models \psi[b, b_1, \ldots, b_n]$. Hence $\mathfrak{B} \models \varphi[b_1, \ldots, b_n]$. $\square$

Theorem 3.5 (Downward Löwenheim–Skolem Theorem). (Uses Choice) *Let $\mathfrak{A}$ be a model for a language $\mathcal{L}$ and let $X \subseteq A$. Then there is a $\mathfrak{B} \prec \mathfrak{A}$ such that $X \subseteq B$ and* $\operatorname{card}(B) \leq \max\{\aleph_0, \operatorname{card}(X), \operatorname{card}(\mathcal{L})\}$.

Proof. Fix a wellordering r of A. For each formula φ of $\mathcal{L}$, let n_φ be 0 if φ is a sentence and let n_φ be the largest number n such that v_n occurs free in φ otherwise. For each $\varphi(v_0, v_1, \ldots, v_{n_\varphi})$, let $f_\varphi : {}^{n_\varphi}A \to A$ be given by

$$f_\varphi(a_1, \ldots, a_{n_\varphi}) = \begin{cases} \text{the } r\text{-least } a \in A \text{ such that } \mathfrak{A} \models \varphi[a, a_1, \ldots, a_{n_\varphi}] \\ \qquad \text{if } \mathfrak{A} \models (\exists v_0)\varphi[a_1, \ldots, a_{n_\varphi}]\,; \\ \text{the } r\text{-least element of } A \text{ otherwise.} \end{cases}$$

(The functions f_φ are called *Skolem functions.*)

Let $Y_0 = X$. For $k \in \omega$ let

$$Y_{k+1} = Y_k \cup \bigcup \{\operatorname{range}(f_\varphi \restriction {}^{n_\varphi}(Y_k)) \mid \varphi \text{ a formula of } \mathcal{L}\}\,.$$

It is easy to prove by induction that $\operatorname{card}(Y_k) \leq \max\{\aleph_0, \operatorname{card}(X), \operatorname{card}(\mathcal{L})\}$ for each $k \in \omega$.

Let $B = \bigcup\{Y_k \mid k \in \omega\}$. Then $\operatorname{card}(B) \leq \max\{\aleph_0, \operatorname{card}(X), \operatorname{card}(\mathcal{L})\}$. Obviously $B \neq \emptyset$. Since B is closed under all the f_φ, it follows that $\mathfrak{A}$ and B satisfy (2) of Theorem 3.4, and so (1) of Theorem 3.4 holds. $\square$

Suppose that $\lhd$ is a linear ordering of a set $I \neq \emptyset$. If $\mathfrak{A}_i$, $i \in I$, are models for a language $\mathcal{L}$ and are such that

$$(\forall i \in I)(\forall j \in I)(i \lhd j \to \mathfrak{A}_i \subseteq \mathfrak{A}_j)\,,$$

then $\langle \triangleleft, \langle \mathfrak{A}_i \mid i \in I \rangle \rangle$ is a *chain of models.* (By $\langle \mathfrak{A}_i \mid i \in I \rangle$ we mean the function f with domain I such that $f(i) = \mathfrak{A}_i$ for each i. We shall often use such notation.)

Let $\langle \triangleleft, \langle \mathfrak{A}_i \mid i \in I \rangle \rangle$ be a chain of models for a language $\mathcal{L}$. Let $A = \bigcup_{i \in I} A_i$ (i.e., $\bigcup\{A_i \mid i \in I\}$). We define a model $\mathfrak{A}$ with universe A as follows:

(i) For constants c of $\mathcal{L}$, set $c_{\mathfrak{A}} = c_{\mathfrak{A}_i}$ for some (all) $i \in I$.

(ii) Let F be a k-place function symbol of $\mathcal{L}$ and let $a_1, \ldots, a_k$ be elements of A. There is an i such that all the a_m belong to A_i. For some (any) such i, set $F_{\mathfrak{A}}(a_1, \ldots, a_k) = F_{\mathfrak{A}_i}(a_1, \ldots, a_k)$.

(iii) Let P be a k-place function symbol of $\mathcal{L}$ and let $a_1, \ldots, a_k$ be elements of A. For some (any) i such that all the a_m belong to A_i, define $P_{\mathfrak{A}}(a_1, \ldots, a_k) \leftrightarrow P_{\mathfrak{A}_i}(a_1, \ldots, a_k)$.

Note that $\mathfrak{A}_i \subseteq \mathfrak{A}$ for each $i \in I$. We call $\mathfrak{A}$ the *union* of the chain of models.

A chain of models $\langle \triangleleft, \langle \mathfrak{A}_i \mid i \in I \rangle \rangle$ is an *elementary chain* if

$$(\forall i \in I)(\forall j \in I)(i \triangleleft j \rightarrow \mathfrak{A}_i \prec \mathfrak{A}_j)\,.$$

Theorem 3.6. *Let $\mathfrak{A}$ be the union of an elementary chain $\langle \triangleleft, \langle \mathfrak{A}_i \mid i \in I \rangle \rangle$. Then $\mathfrak{A}_i \prec \mathfrak{A}$ for every $i \in I$.*

Proof. By induction on the complexity of formulas $\varphi(x_1, \ldots, x_n)$, we show that

$$(\forall i \in I)(\forall a_1, \ldots, a_n \in A_i)(\mathfrak{A}_i \models \varphi[a_1, \ldots, a_n] \leftrightarrow \mathfrak{A} \models \varphi[a_1, \ldots, a_n])\,.$$

The cases that φ is atomic, that it is a negation, and that it is a conjunction are routine.

For the case that φ is $(\exists y)\psi$ for some formula $\psi(y, x_1, \ldots, x_n)$, let $i \in I$ and assume first that $\mathfrak{A}_i \models \varphi[a_1, \ldots, a_n]$. Then there is a $b \in A_i$ such that $\mathfrak{A}_i \models \psi[b, a_1, \ldots, a_n]$. By induction, $\mathfrak{A} \models \psi[b, a_1, \ldots, a_n]$ and hence $\mathfrak{A} \models \varphi[a_1, \ldots, a_n]$.

Now suppose that $\mathfrak{A} \models \varphi[a_1, \ldots, a_n]$. Let $b \in A$ be such that $\mathfrak{A} \models \psi[b, a_1, \ldots, a_n]$. There is a $j \in I$ with $i \triangleleft j$ or $i = j$ and such that $b \in A_j$. By induction we get that $\mathfrak{A}_j \models \psi[b, a_1, \ldots, a_n]$ and so that $\mathfrak{A}_j \models \varphi[a_1, \ldots, a_n]$. Since $\mathfrak{A}_i \prec \mathfrak{A}_j$, it follows that $\mathfrak{A}_i \models \varphi[a_1, \ldots, a_n]$. □

Exercise 3.1. Let $(\omega; <)$ be the obvious model for the language $\{<\}$. (We shall frequently specify models in this way.) For models $\mathfrak{A}$ of $\mathrm{Th}(\omega; <)$—we omit a set of parentheses for appearance's sake—and elements a and b of A, say that $a \sim_{\mathfrak{A}} b$ if there are only finitely many elements of A that are between a and b with respect to $<_{\mathfrak{A}}$. Let $[a]_{\mathfrak{A}}$ be the equivalence class of $a \in A$ with respect to $\sim_{\mathfrak{A}}$. Say that $[a]_{\mathfrak{A}} \ll_{\mathfrak{A}} [b]_{\mathfrak{A}}$ if $a <_{\mathfrak{A}} b$ and there are infinitely many c such that $a <_{\mathfrak{A}} c <_{\mathfrak{A}} b$. Use elementary chains to show that there is a model $\mathfrak{B}$ of $\mathrm{Th}(\omega; <)$ such that $\ll_{\mathfrak{B}}$ is a dense linear ordering of the set of equivalence classes of $\sim_{\mathfrak{B}}$ with no last element.

Hint. First show that, for any model $\mathfrak{A}$ of $\mathrm{Th}(\omega; <)$, there is an elementary extension $\mathfrak{C}$ of $\mathfrak{A}$ with the following properies:

(1) $(\forall a \in A)(\exists c \in C)\, [a]_{\mathfrak{C}} \ll_{\mathfrak{C}} [c]_{\mathfrak{C}}$;

(2) $(\forall a \in A)(\forall a' \in A)([a]_{\mathfrak{A}} \ll_{\mathfrak{A}} [a']_{\mathfrak{A}} \to (\exists c \in C)\, [a]_{\mathfrak{C}} \ll_{\mathfrak{C}} [c]_{\mathfrak{C}} \ll_{\mathfrak{C}} [a']_{\mathfrak{C}})$.

(Note that $[a]_{\mathfrak{C}} = [a]_{\mathfrak{A}}$ for $a \in A$.) To show that $\mathfrak{C}$ exists, use compactness and Theorem 3.1. Your expanded language could have one or infinitely many constants for each instance of (2), though one constant suffices to take care of (1).

If $\mathfrak{A}$ and $\mathfrak{B}$ are models for a language $\mathcal{L}$, f is an *elementary embedding* of $\mathfrak{A}$ into $\mathfrak{B}$ ($f : \mathfrak{A} \prec \mathfrak{B}$ or $\mathfrak{A} \overset{f}{\prec} \mathfrak{B}$) if, for all formulas $\varphi(x_1, \ldots, x_n)$ of $\mathcal{L}$ and any elements $a_1, \ldots, a_n$ of A, $\mathfrak{A} \models \varphi[a_1, \ldots, a_n]$ if and only if $\mathfrak{B} \models \varphi[f(a_1), \ldots, f(a_n)]$.

A *theory* in a language $\mathcal{L}$ is a set Σ of sentences such that whenever $\Sigma \models \tau$ then $\tau \in \Sigma$. A theory in $\mathcal{L}$ is *complete* if it is consistent and, for every sentence τ of $\mathcal{L}$, either τ or $\neg\tau$ belongs to Σ.

Theorem 3.7 (Robinson Joint Consistency Theorem). (Uses Choice.) *Let $\mathcal{L}'$ and $\mathcal{L}''$ be languages and let $\mathcal{L} = \mathcal{L}' \cap \mathcal{L}''$. Let T' be a consistent theory in $\mathcal{L}'$. Let T'' be a consistent theory in $\mathcal{L}''$. Let T be a complete theory in $\mathcal{L}$ such that $T \subseteq T' \cap T''$. Then $T' \cup T''$ is consistent.*

Remark. The statement of the theorem is somewhat imprecise. By saying that $\mathcal{L} = \mathcal{L}' \cap \mathcal{L}''$ we mean to imply that the only common symbols of $\mathcal{L}'$ and $\mathcal{L}''$ are those of $\mathcal{L}$, and that each of these common symbols is the same kind of symbol in the three languages.

Proof. In order to do an elementary chain construction, we need the following two lemmas.

Lemma 3.8. (Uses Choice.) *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be languages each of which extends a language $\mathcal{L}_0$. Let $\mathfrak{A}$ and $\mathfrak{B}$ be models for $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively. Suppose that*

$$\mathrm{Th}(\mathfrak{A} \restriction \mathcal{L}_0) = \mathrm{Th}(\mathfrak{B} \restriction \mathcal{L}_0)\,,$$

where, e.g., $\mathfrak{A} \restriction \mathcal{L}_0$ is the reduct of $\mathfrak{A}$ to $\mathcal{L}_0$. Then there is a model $\mathfrak{A}^*$ for $\mathcal{L}_1$ and there is a function g such that

(i) $\mathfrak{A} \prec \mathfrak{A}^*$*;*

(ii) $g : \mathfrak{B} \restriction \mathcal{L}_0 \prec \mathfrak{A}^* \restriction \mathcal{L}_0$*.*

Proof. We may assume that the constants of $(\mathcal{L}_1)_A \setminus \mathcal{L}_1$ are not symbols of $(\mathcal{L}_2)_B$ and vice versa. We may also assume that $(\mathcal{L}_0)_B$ and $(\mathcal{L}_2)_B$ have the same constants c^b.

We first show that $\mathrm{Th}((\mathfrak{B} \restriction \mathcal{L}_0)_B) \cup \mathrm{Th}(\mathfrak{A}_A)$ is consistent. Assume otherwise. Using compactness and forming conjunctions, we get that there are sentences $\sigma \in \mathrm{Th}((\mathfrak{B} \restriction \mathcal{L}_0)_B)$ and $\tau \in \mathrm{Th}((\mathfrak{A})_A)$ such that $\{\sigma, \tau\}$ is inconsistent. Hence $\sigma \models \neg\tau$. There is a formula $\varphi(x_1, \ldots, x_n)$ of $\mathcal{L}_0$ such that σ is $\varphi(c_1, \ldots, c_n)$ for constants $c_1, \ldots, c_n$ of $(\mathcal{L}_0)_B \setminus \mathcal{L}_0$. Hence $\varphi(c_1, \ldots, c_n) \models \neg\tau$. Since the constants $c_1, \ldots, c_n$ are not constants of $(\mathcal{L}_1)_A$, we can apply n times property (XII) of $\models$ and get that

$$(\exists x_1) \cdots (\exists x_n)\, \varphi(x_1, \ldots, x_n) \models \neg\tau\,.$$

The sentence $(\exists x_1) \cdots (\exists x_n)\, \varphi(x_1, \ldots, x_n)$ belongs to $\mathrm{Th}(\mathfrak{B} \restriction \mathcal{L}_0)$. By hypothesis it must then belong to $\mathrm{Th}(\mathfrak{A} \restriction \mathcal{L}_0)$, and so to $\mathrm{Th}(\mathfrak{A}_A)$. This contradicts the fact that it implies $\neg\tau$.

Let $\mathfrak{C}$ be a model of $\mathrm{Th}((\mathfrak{B} \restriction \mathcal{L}_0)_B) \cup \mathrm{Th}(\mathfrak{A}_A)$. By Theorem 3.1, there is a model $\mathfrak{A}^*$ for $\mathcal{L}_1$ such that

$$\mathfrak{A} \prec \mathfrak{A}^* \cong \mathfrak{C} \restriction \mathcal{L}_1\,.$$

The function $b \mapsto c^b_{\mathfrak{C}}$ is an elementary embedding of $\mathfrak{B} \restriction \mathcal{L}_0$ into $\mathfrak{C} \restriction \mathcal{L}_0 \cong \mathfrak{A}^* \restriction \mathcal{L}_0$, so we get a g satisfying (ii). $\square$

Lemma 3.9. (Uses Choice). *Let $\mathcal{L}_0$, $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathfrak{A}$, and $\mathfrak{B}$ be as in Lemma 3.8. In addition, let*

$$f : \mathfrak{A} \restriction \mathcal{L}_0 \prec \mathfrak{B} \restriction \mathcal{L}_0\,.$$

Then there are $\mathfrak{A}^*$ and g satisfying (i) and (ii) of Lemma 3.8 and such that

(iii) $g \circ f$ is the identity.

Proof. We assume that $(\mathcal{L}_0)_A$ and $(\mathcal{L}_1)_A$ have the same constants c^a, and we let $(\mathcal{L}_2)_A$ be the extension of $\mathcal{L}_2$ with these same constants, which we assume are not symbols of $\mathcal{L}_2$. Let $\hat{\mathfrak{B}}$ be the expansion of $\mathfrak{B}$ to $(\mathcal{L}_2)_A$ gotten by setting $c^a_{\hat{\mathfrak{B}}} = f(a)$ for each $a \in A$. The hypotheses of Lemma 3.8 are satisfied by the languages $(\mathcal{L}_0)_A$, $(\mathcal{L}_1)_A$, and $(\mathcal{L}_2)_A$ and the models $\mathfrak{A}_A$ and $\hat{\mathfrak{B}}$. By that lemma, we get $\tilde{\mathfrak{A}}$ and g such that

(1) $\mathfrak{A}_A \prec \tilde{\mathfrak{A}}$;

(2) $g : \hat{\mathfrak{B}} \restriction (\mathcal{L}_0)_A \prec \tilde{\mathfrak{A}} \restriction (\mathcal{L}_0)_A$.

Let $\mathfrak{A}^* = \tilde{\mathfrak{A}} \restriction \mathcal{L}_1$. Clause (i) follows from (1), and clause (ii) follows from (2). (Note that g is literally a function with domain B.) For (iii), let $a \in A$. Then

$$g(f(a)) = g(c^a_{\hat{\mathfrak{B}}}) = c^a_{\tilde{\mathfrak{A}}} = c^a_{\mathfrak{A}_A} = a\,. \qquad \square$$

Let $\mathfrak{A}_0$ and $\mathfrak{B}_0$ be models of T' and T'' respectively. Applying Lemma 3.8 with languages $\mathcal{L}$, $\mathcal{L}''$, and $\mathcal{L}'$ and models $\mathfrak{B}_0$ and $\mathfrak{A}_0$, we get $\mathfrak{B}_1 \succ \mathfrak{B}_0$ and $f_0 : \mathfrak{A}_0 \restriction \mathcal{L} \prec \mathfrak{B}_1 \restriction \mathcal{L}$.

Suppose inductively that we have (1) a model $\mathfrak{A}_n$ of T', (2) a model $\mathfrak{B}_{n+1}$ of T'', and (3) an elementary embedding $f_n : \mathfrak{A}_n \restriction \mathcal{L} \prec \mathfrak{B}_{n+1} \restriction \mathcal{L}$. Applying Lemma 3.9 with languages $\mathcal{L}$, $\mathcal{L}'$, and $\mathcal{L}''$ and models $\mathfrak{A}_n$ and $\mathfrak{B}_{n+1}$, we get $\mathfrak{A}_{n+1} \succ \mathfrak{A}_n$ and $g_{n+1} : \mathfrak{B}_{n+1} \restriction \mathcal{L} \prec \mathfrak{A}_{n+1} \restriction \mathcal{L}$, such that $g_{n+1} \circ f_n$ is the identity. By another application of Lemma 3.9, we get $\mathfrak{B}_{n+2} \succ \mathfrak{B}_{n+1}$ and $f_{n+1} : \mathfrak{A}_{n+1} \restriction \mathcal{L} \prec \mathfrak{B}_{n+2} \restriction \mathcal{L}$, such that $f_{n+1} \circ g_{n+1}$ is the identity.

Both $\langle <, \langle \mathfrak{A}_n \mid n \in \omega \rangle \rangle$ and $\langle <, \langle \mathfrak{B}_n \mid n \in \omega \rangle \rangle$ are elementary chains. Moreover, for each $n \in \omega$,

$$f_n = (f_{n+1} \circ g_{n+1}) \circ f_n = f_{n+1} \circ (g_{n+1} \circ f_n) = f_{n+1} \restriction A_n\,.$$

Similarly $g_{n+1} = g_{n+2} \restriction B_{n+1}$ for each $n \in \omega$.

Let $\mathfrak{A}$ and $\mathfrak{B}$ the unions of the elementary chains $\langle <, \langle \mathfrak{A}_n \mid n \in \omega \rangle \rangle$ and $\langle <, \langle \mathfrak{B}_n \mid n \in \omega \rangle \rangle$. By Theorem 3.6, $\mathfrak{A} \models T'$ and $\mathfrak{B} \models T''$.

Let $f : A \to B$ be given by $f = \bigcup_{n \in \omega} f_n$ and let $g : B \to A$ be given by $g = \bigcup_{n \in \omega} g_n$. It is easy to see that f and g are inverses of one another and that $f : \mathfrak{A} \restriction \mathcal{L} \cong \mathfrak{B} \restriction \mathcal{L}$. Define an expansion $\mathfrak{C}$ of $\mathfrak{A}$ to $\mathcal{L}' \cup \mathcal{L}''$ by making $f : \mathfrak{C} \restriction \mathcal{L}'' \cong \mathfrak{B}$. The model $\mathfrak{C}$ witnesses that $T' \cup T''$ is consistent. $\square$

Corollary 3.10 (Craig's Lemma). *Let σ and τ be sentences of some language such that $\sigma \models \tau$. Then there is a sentence θ of the language such that every constant, function symbol, and relation symbol occurring in θ occurs in both σ and τ and such that $\sigma \models \theta$ and $\theta \models \tau$.*

Proof. Let the non-logical symbols of $\mathcal{L}$ be those occurring in both σ and τ. Let the non-logical symbols of $\mathcal{L}'$ be those occurring in σ, and let the non-logical symbols of $\mathcal{L}''$ be those occurring in τ. Let T_0 be the set of all sentences θ of $\mathcal{L}$ such that $\sigma \models \theta$. If $T_0 \models \tau$, then we get the desired θ by compactness. Let then $\mathfrak{A}$ be a model for $\mathcal{L}''$ such that $\mathfrak{A} \models T_0 \cup \{\neg\tau\}$. Let $T'' = \mathrm{Th}(\mathfrak{A})$ and let $T = \mathrm{Th}(\mathfrak{A} \restriction \mathcal{L})$. Let T' be the set of consequences in $\mathcal{L}'$ of $T \cup \{\sigma\}$. If T' were inconsistent, then compactness would give a $\theta \in T$ such that $\theta \models \neg\sigma$. This would yield the contradiction that $\sigma \models \neg\theta$ and so that $\neg\theta \in T_0 \subseteq T$. Thus the hypotheses of Theorem 3.7 are satisfied. By that theorem, $T' \cup T''$ is consistent, contradicting the assumption that $\sigma \models \tau$. $\square$

Let $\mathcal{L}$ be a language, and let $\mathcal{L} \cup \{P, P'\}$ be the result of adding to $\mathcal{L}$ new k-place relation symbols P and P'. Let $\Sigma(P)$ be a set of sentences of $\mathcal{L} \cup \{P\}$ and let $\Sigma(P')$ result from $\Sigma(P)$ by replacing each occurrence of P by an occurrence of P'.

We say that $\Sigma(P)$ *defines P implicitly* if

$$\Sigma(P) \cup \Sigma(P') \models (\forall x_1) \cdots (\forall x_k)(P(x_1, \ldots, x_k) \leftrightarrow P'(x_1, \ldots, x_k)) .$$

(In other words, if $\mathfrak{A}$ is a model for $\mathcal{L}$, then there is at most on way to expand $\mathfrak{A}$ to a model of $\Sigma(P)$.)

We say that $\Sigma(P)$ *defines P explicitly* if there is a formula $\varphi(x_1, \ldots, x_k)$ of $\mathcal{L}$ such that

$$\Sigma(P) \models (\forall x_1) \cdots (\forall x_k)(P(x_1, \ldots, x_k) \leftrightarrow \varphi(x_1, \ldots, x_k)) .$$

Theorem 3.11 (Beth's Theorem). (Uses Choice.) *$\Sigma(P)$ defines P implicitly if and only if $\Sigma(P)$ defines P explicitly.*

Proof. The "if" part of the theorem is obvious. For the "only if" part, assume that $\Sigma(P)$ defines P implicitly.

Adjoin new constants $c_1, \ldots, c_k$ to $\mathcal{L}$. We have that

$$\Sigma(P) \cup \Sigma(P') \models P(c_1, \ldots, c_k) \to P'(c_1, \ldots, c_k) .$$

By compactness, we get a finite $\Delta \subseteq \Sigma(P)$ and a finite $\Delta' \subseteq \Sigma(P')$ such that

$$\Delta \cup \Delta' \models P(c_1, \ldots, c_k) \to P'(c_1, \ldots, c_k) .$$

We may assume without loss of generality that Δ' is the set of sentences that result from Δ when all occurrences of P are replaced by occurrences of P'.

Let $\sigma(P)$ be the conjunction of all the members of Δ and let $\sigma(P')$ be the conjunction of all the members of Δ'. Then

$$\sigma(P) \wedge \sigma(P') \models P(c_1, \dots, c_k) \rightarrow P'(c_1, \dots, c_k) .$$

From this it follows that

$$\sigma(P) \wedge P(c_1, \dots, c_k) \models \sigma(P') \rightarrow P'(c_1, \dots, c_k) .$$

By Craig's Lemma there is a sentence $\theta(c_1, \dots, c_k)$ of $\mathcal{L} \cup \{c_1, \dots, c_k\}$ such that

(a) $\sigma(P) \wedge P(c_1, \dots, c_k) \models \theta(c_1, \dots, c_k)$;
(b) $\theta(c_1, \dots, c_k) \models \sigma(P') \rightarrow P'(c_1, \dots, c_k)$.

From (b) we it follows that

$$\theta(c_1, \dots c_k) \models \sigma(P) \rightarrow P(c_1, \dots, c_k) ,$$

and so that

$$\sigma(P) \models \theta(c_1, \dots c_k) \rightarrow P(c_1, \dots c_k) .$$

But (a) implies that

$$\sigma(P) \models P(c_1, \dots, c_k) \rightarrow \theta(c_1, \dots, c_k) .$$

Hence

$$\sigma(P) \models P(c_1, \dots, c_k) \leftrightarrow \theta(c_1, \dots, c_k) .$$

Since $c_1, \dots, c_k$ do not occur in $\sigma(P)$,

$$\sigma(P) \models (\forall x_1) \cdots (\forall x_k)(P(x_1, \dots, x_k) \leftrightarrow \theta(x_1, \dots, x_k)) .$$

Since $\Sigma(P) \models \sigma(P)$, the proof is complete. $\square$

Exercise 3.2. Prove the Robinson Joint Consistency Theorem directly from Craig's Lemma.

Exercise 3.3. A model a for a language $\mathcal{L}$ is *finitely generated* if there is a finite $X \subseteq A$ such that there is no $\mathfrak{B} \subsetneq \mathfrak{A}$ with $X \subseteq B$. Let T be a theory in $\mathcal{L}$ and let $\mathfrak{A}$ be a model for $\mathcal{L}$. Assume that every finitely generated submodel of $\mathfrak{A}$ is isomorphic to a submodel of a model of T. Show that $\mathfrak{A}$ is isomorphic to a submodel of a model of T.

Hint. First prove an analogue of Theorem 3.1 for $\subseteq$.

Exercise 3.4. Let $\mathcal{L}$ be a language containing a two-place relation symbol R. If $\mathfrak{A}$ is a model for $\mathcal{L}$, then an *end extension* of $\mathfrak{A}$ (with respect to R) is a $\mathfrak{B} \supsetneq \mathfrak{A}$ such that

$$(\forall a \in A)(\forall a \in B \setminus A)(R_{\mathfrak{B}}(a, b) \wedge \neg R_{\mathfrak{B}}(b, a)) .$$

Let T be a theory in $\mathcal{L}$ and suppose that every countable model of T has an elementary end extension. Show that every countable model of T has an uncountable elementary end extension.

Assume until further notice that $\mathcal{L}$ is a countable language.

For $n \in \omega$, an *n-type* (in $\mathcal{L}$) is a set $\Sigma(x_1, \ldots, x_n)$ of formulas with only the (distinct) variables $x_1, \ldots, x_n$ free and such that

(1) If new constants $c_1, \ldots, c_n$ are adjoined to $\mathcal{L}$, then $\{\varphi(c_1, \ldots, c_n) \mid \varphi(x_1, \ldots, x_n) \in \Sigma(x_1, \ldots, x_n)\}$ is consistent.

(2) If φ is a formula of $\mathcal{L}$ with only $x_1, \ldots, x_n$ free, then either $\varphi \in \Sigma(x_1, \ldots, x_n)$ or $\neg\varphi \in \Sigma(x_1, \ldots, x_n)$.

A 0-type is just a complete theory.

If T is a theory in $\mathcal{L}$, an *n-type of T* is n-type of which T is a subset.

A model $\mathfrak{A}$ for $\mathcal{L}$ *realizes* an n-type $\Sigma(x_1, \ldots, x_n)$ if there are elements $a_1, \ldots, a_n$ of A such that

$$(\forall \varphi \in \Sigma(x_1, \ldots, x_n))\, \mathfrak{A} \models \varphi[a_1, \ldots, a_n] .$$

If $\mathfrak{A}$ does not realize $\Sigma(x_1, \ldots, x_n)$, then we say that $\mathfrak{A}$ *omits* $\Sigma(x_1, \ldots, x_n)$.

If $\mathfrak{A}$ is a model for $\mathcal{L}$ and $Y \subseteq A$, let $\mathcal{L}_Y$ come from $\mathcal{L}$ by adding the new constants c^a for $a \in Y$. Let $\mathfrak{A}_Y$ be the obvious expansion of $\mathfrak{A}$ to $\mathcal{L}_Y$.

For infinite cardinal numbers κ, a model $\mathfrak{A}$ is *κ-saturated* if, for every $Y \subseteq A$ with $|Y| < \kappa$, $\mathfrak{A}_Y$ realizes every one-type $\Sigma(x)$ of $\mathrm{Th}(\mathfrak{A}_Y)$. If $\mathfrak{A}$ is $|A|$-saturated, then $\mathfrak{A}$ is *saturated.* If $\mathfrak{A}$ is both countable and ω-saturated (i.e. $\aleph_0$-saturated), then $\mathfrak{A}$ is *countably saturated.*

Theorem 3.12. *Let T be a complete theory in $\mathcal{L}$. Then T has a countably saturated model if and only if, for all $n \in \omega$, T has only countably many n-types.*

Proof. Suppose that $\mathfrak{A}$ is an ω-saturated model of T. We show by induction on n that every n-type of T is realized in $\mathfrak{A}$. The case $n = 0$ is trivial. (So is the case $n = 1$: take $Y = \emptyset$.)

Let $n \in \omega$ and let $\Sigma(x_1, \ldots, x_{n+1})$ be an $n{+}1$-type of T. Let $\bar{\Sigma}(x_1, \ldots, x_n) \subseteq \Sigma(x_1, \ldots, x_{n+1})$ be the corresponding n-type of T. By induction, let $a_1, \ldots, a_n$ be elements of A such that

$$(\forall \varphi(x_1, \ldots, x_n) \in \bar{\Sigma}(x_1, \ldots, x_n))\, \mathfrak{A} \models \varphi[a_1, \ldots, a_n]\,.$$

Let $Y = \{a_1, \ldots, a_n\}$. Let

$$\Sigma^*(x_{n+1}) = \{\varphi(c^{a_1}, \ldots, c^{a_n}, x_{n+1}) \mid \varphi(x_1, \ldots, x_{n+1}) \in \Sigma(x_1, \ldots, x_{n+1})\}\,.$$

If c is a new constant, then

$$\begin{aligned}&\{\psi(c) \mid \psi(x_{n+1}) \in \Sigma^*(x_{n+1})\}\\ &\quad = \{\varphi(c^{a_1}, \ldots, c^{a_n}, c) \mid \varphi(x_1, \ldots, x_{n+1}) \in \Sigma(x_1, \ldots, x_{n+1})\}\,,\end{aligned}$$

and so this set is consistent. The set $\Sigma^*(x_{n+1})$ fails to satisfy requirement (2) for being a type in $\mathcal{L}_Y$, but it satisfies (2) except for formulas φ of $\mathcal{L}_Y$ that contain an occurrence of some c^{a_i} within the scope of a quantifier containing the corresponding x_i. Moreover $\Sigma^*(x_{n+1})$ fails only for the same trivial reason to include $\mathrm{Th}(\mathfrak{A}_Y)$. Thus there is a one-type $\Sigma^{**}(x_{n+1})$ of $\mathrm{Th}(\mathfrak{A}_Y)$ such that $\Sigma^*(x_{n+1}) \subseteq \Sigma^{**}(x_{n+1})$. By ω-saturation, let $a_{n+1} \in A$ be such that $\mathfrak{A}_Y \models \psi[a_{n+1}]$ for all $\psi(x_{n+1}) \in \Sigma^{**}(x_{n+1})$. Thus $\mathfrak{A} \models \varphi[a_1, \ldots, a_{n+1}]$ for all $\varphi(x_1, \ldots, x_{n+1}) \in \Sigma(x_1, \ldots, x_{n+1})$, so $\mathfrak{A}$ realizes $\Sigma(x_1, \ldots, x_{n+1})$.

To see that what we have proved implies the "only if" part of the theorem, suppose that $\mathfrak{A}$ is a countably saturated model of T. Since ${}^nA$ is countable, $\mathfrak{A}$ realizes only countably many n-types. But these are all the n-types of T.

For the "if" part of the theorem, first let $\mathfrak{A}$ be a countable model of T. (Recall that $\mathcal{L}$ is countable.) We show that there is a countable $\mathfrak{B}$ such that $\mathfrak{A} \prec \mathfrak{B}$ and, for all finite $Y \subseteq A$ and all one-types $\Sigma(x)$ of $\mathrm{Th}(\mathfrak{A}_Y)$, $\mathfrak{B}_Y$ realizes $\Sigma(x)$. Starting with some countable model $\mathfrak{A}_0$ of T and repeatedly applying this lemma, we get

$$\mathfrak{A}_0 \prec \mathfrak{A}_1 \prec \mathfrak{A}_2 \prec \cdots\,,$$

such that each $(\mathfrak{A}_{i+1})_Y$ is countable and realizes each one-type of $(\mathfrak{A}_i)_Y$ for each finite $Y \subseteq A_i$. The union of this elementary chain is thus countable and ω-saturated.

To show that $\mathfrak{B}$ exists, let Let

$$W = \{\langle Y, \Sigma(x)\rangle \mid Y \subseteq A \ \wedge\ Y \text{ is finite } \wedge\ \Sigma(x) \text{ is a one-type of } \mathrm{Th}(\mathfrak{A}_Y)\}$$

and let $\mathcal{L}^*$ be the language

$$\mathcal{L}_A \cup \{d^{Y,\Sigma(x)} \mid \langle Y, \Sigma(x)\rangle \in W\},$$

where the $d^{Y,\Sigma(x)}$ are new constants. Since distinct one-types of $\mathrm{Th}(\mathfrak{A}_Y)$ give rise to distinct $(|Y|+1)$-types of T, we know that each $\mathrm{Th}(\mathfrak{A}_Y)$ has only countably many one-types. Thus $\mathcal{L}^*$ is countable.

Let

$$\Gamma = \mathrm{Th}(\mathfrak{A}_A) \cup \{\varphi(d^{Y,\Sigma(x)}) \mid \langle Y, \Sigma(x)\rangle \in W \wedge \varphi(x) \in \Sigma(x)\}.$$

Every finite subset of Γ is satisfiable in an expansion of $\mathfrak{A}_A$. To see this, let $\langle Y, \Sigma(x)\rangle \in W$. Then note that, for any finite conjunction $\varphi(x)$ of members of $\Sigma(x)$, the sentence $(\exists x)\,\varphi(x) \in \mathrm{Th}(\mathfrak{A}_Y) \subseteq \mathrm{Th}(\mathfrak{A}_A)$. Thus compactness gives a countable model $\mathfrak{B}^*$ of Γ. Using Theorem 3.1, let $\mathfrak{B}$ be isomorphic to the reduct of $\mathfrak{B}^*$ to $\mathcal{L}$ and such that $\mathfrak{A} \prec \mathfrak{B}$. $\square$

Exercise 3.5. Prove that a model $\mathfrak{A}$ is κ-saturated if and only if, for every $n \in \omega$ and every $Y \subseteq A$ with $|Y| < \kappa$, $\mathfrak{A}_Y$ realizes every n-type $\Sigma(x)$ of $\mathrm{Th}(\mathfrak{A}_Y)$. (You don't have to give the detailed argument.)

Exercise 3.6. Let T be a complete theory and suppose that all countable models of T are ω-saturated. Show that all models of T are ω-saturated.

Exercise 3.7. A set x is *hereditarily countable* if the transitive closure of x is countable. Let $\mathfrak{A}$ be the model for the language of set theory which has the set of all hereditarily countable sets as its universe and which is such that $\in_{\mathfrak{A}} = \in \restriction A$. Prove that $\mathfrak{A}$ realizes uncountably many one-types of $\mathrm{Th}(\mathfrak{A})$.

Hint. The set $\mathcal{P}(\omega)$ is uncountable.

Theorem 3.13. *Let T be a complete theory in $\mathcal{L}$. Any two countably saturated models of T are isomorphic.*

Proof. Let $\mathfrak{A}$ and $\mathfrak{B}$ be countably saturated models of T. Let $Y \subseteq A$ and $Z \subseteq B$ be such that Y and Z are finite. Let $Y = \{a_1, \ldots, a_n\}$. Suppose that $f : Y \to Z$ is one-one onto and is such that, for all formulas $\varphi(x_1, \ldots, x_n)$ of $\mathcal{L}$,

$$\mathfrak{A} \models \varphi[a_1, \ldots, a_n] \;\leftrightarrow\; \mathfrak{B} \models \varphi[f(a_1), \ldots, f(a_n)].$$

Let $a \in A$. We show that there is a $b \in B$ such that, for all formulas $\varphi(x_1, \ldots, x_{n+1})$ of $\mathcal{L}$,

$$\mathfrak{A} \models \varphi[a_1, \ldots, a_n, a] \;\leftrightarrow\; \mathfrak{B} \models \varphi[f(a_1), \ldots, f(a_n), b].$$

To show this, let $\Sigma(x_{n+1})$ be the one-type of $\mathrm{Th}(\mathfrak{A}_Y)$ given by a. Let

$$\tilde{\Sigma}(x_{n+1}) = \{\varphi(c^{f(a_1)}, \ldots, c^{f(a_n)}, x_{n+1}) \mid \varphi(c^{a_1}, \ldots, c^{a_n}, x_{n+1}) \in \Sigma(x_{n+1})\} .$$

Clearly $\tilde{\Sigma}(x_{n+1})$ is a one-type of $\mathrm{Th}(\mathfrak{B}_Z)$. By saturation, $\mathfrak{B}_Z$ realizes this one-type. Let b witness this fact.

We can prove by the same method that for every $b \in B$ there is an $a \in A$ such that, for all formulas $\varphi(x_1, \ldots, x_{n+1})$ of $\mathcal{L}$,

$$\mathfrak{A} \models \varphi[a_1, \ldots, a_n, a] \;\leftrightarrow\; \mathfrak{B} \models \varphi[f(a_1), \ldots, f(a_n), b] .$$

Since A and B are countable, these facts allow us, starting with the empty $f : \emptyset \to \emptyset$, to construct by recursion an isomorphism $g : \mathfrak{A} \cong \mathfrak{B}$. $\square$

Theorem 3.14. *Let T be a complete theory. Any two saturated models of T of the same cardinality are isomorphic.*

Proof. The proof is a direct generalization of the proof of Theorem 3.13, so we omit it. $\square$

Let T be a theory (in $\mathcal{L}$). A type $\Sigma(x_1, \ldots, x_n)$ of T is *principal* if there is a finite $\Delta(x_1, \ldots, x_n) \subseteq \Sigma(x_1, \ldots, x_n)$ such that, for all $\psi(x_1, \ldots, x_n) \in \Sigma(x_1, \ldots, x_n)$,

$$(\forall x_1) \cdots (\forall x_n)(\bigwedge\!\!\!\bigwedge \Delta(x_1, \ldots, x_n) \to \psi(x_1, \ldots, x_n)) \in T ,$$

where $\bigwedge\!\!\!\bigwedge \Delta(x_1, \ldots, x_n)$ is the conjunction of all the formulas belonging to $\Delta(x_1, \ldots, x_n)$.

Theorem 3.15. *Let T be a theory in $\mathcal{L}$ and let $n \in \omega$. The following are equivalent:*

(a) There is a non-principal n-type of T.

(b) There are infinitely many n-types of T (for fixed $x_1, \ldots, x_n$).

Proof. To show that (a) implies (b), let $\Sigma(x_1, \ldots, x_n)$ be a non-principal n-type of T. Let $k \in \omega$ and assume that

$$\Sigma_0(x_1, \ldots, x_n), \ldots, \Sigma_{k-1}(x_1, \ldots, x_n)$$

are the only n-types of T that are distinct from $\Sigma(x_1, \ldots, x_n)$. For each $i < k$ let

$$\varphi_i(x_1, \ldots, x_n) \in \Sigma(x_1, \ldots, x_n) \setminus \Sigma_i(x_1, \ldots, x_n) .$$

Let $\Delta(x_1, \dots, x_n) = \{\varphi_i \mid i < k\}$. Since $\Sigma(x_1, \dots, x_n)$ is non-principal, there is a $\psi(x_1, \dots, x_n) \in \Sigma(x_1, \dots, x_n)$ such that

$$\Gamma \;=\; T \cup \{(\exists x_1) \cdots (\exists x_n)(\bigwedge\!\!\!\bigwedge \Delta(x_1, \dots, x_n) \;\wedge\; \neg\psi(x_1, \dots, x_n))\}$$

is consistent. Let $\mathfrak{A}$ be a model of Γ. Let $a_1, \dots, a_n$ be members of A such that

$$\mathfrak{A} \;\models\; (\bigwedge\!\!\!\bigwedge \Delta(x_1, \dots, x_n) \;\wedge\; \neg\psi(x_1, \dots, x_n))[a_1, \dots, a_n].$$

Then

$$\{\varphi(x_1, \dots, x_n) \mid \mathfrak{A} \models \varphi[a_1, \dots, a_n]\}$$

is an n-type of T that is distinct from $\Sigma(x_1, \dots, x_n)$ and from each of the $\Sigma_i(x_1, \dots, x_n)$.

For the implication from (b) to (a), let φ_i, $i \in \omega$, be all formulas of $\mathcal{L}$ with only $x_1, \dots, x_n$ free. Let $k \in \omega$ and assume inductively that, for each $i < k$, ψ_i is either φ_i or $\neg\varphi_i$. Also assume inductively that there are infinitely many n-types of T that include $\{\psi_i \mid i < k\}$. Obviously there is a choice of φ_k that satisfies our induction hypotheses for $k + 1$. Thus we get $\{\psi_i \mid i \in \omega\}$, an n-type of T. If $\Delta(x_1, \dots, x_n)$ witnessed that this type were principal, then there would be a k with $\Delta(x_1, \dots, x_n) \subseteq \{\psi_i \mid i < k\}$. But infinitely many n-types of T include this set. $\square$

Theorem 3.16. *Let T be a theory in $\mathcal{L}$ and let $\Sigma(x_1, \dots, x_n)$ be a non-principal type of T. Then T has a countable model that omits $\Sigma(x_1, \dots, x_n)$.*

Proof. Let c_i, $i \in \omega$, be new constants. Let $\mathcal{L}^* = \mathcal{L} \cup \{c_i \mid i \in \omega\}$. We shall construct a theory $T^* \supseteq T$ in $\mathcal{L}^*$ such that

(1) T^* is consistent;

(2) if $(\exists x)\varphi(x) \in T^*$ then $\varphi(c_i) \in T^*$ for some $i \in \omega$;

(3) T^* is complete;

(4) for all $i_1, \dots, i_n \in \omega$, there is a $\varphi(x_1, \dots, x_n) \in \Sigma(x_1, \dots, x_n)$ such that $\varphi(c_{i_1}, \dots, c_{i_n}) \notin T^*$.

The existence of such a T^* suffices to prove the theorem, as the following argument shows. The proof of Theorem 2.1, with T^* for the Σ of that theorem, gives a countable model $\mathfrak{A}^*$ of T^* such that

$$A^* = \{c_{\mathfrak{A}^*} \mid c \text{ is a constant of } \mathcal{L}^*\}\,.$$

For any constant c of $\mathcal{L}^*$, an application of (2) with $c = x$ for $\varphi(x)$ shows that every there is an i such that $c = c_i \in T^*$. Thus

$$A^* = \{c_{i\mathfrak{A}^*} \mid i \in \omega\}\,.$$

This and (4) imply that the reduct of $\mathfrak{A}^*$ to $\mathcal{L}$ omits Σ.

To construct T^*, we follow the proof of Theorem 2.8, with an extra step to take care of (4).

Let $\sigma_i,\ i \in \omega$, be all sentences of $\mathcal{L}^*$ Let $\langle (i)_1, \ldots, (i)_n\rangle,\ i \in \omega$, be all elements of ${}^n\omega$.

By recursion, we define sets Σ_α of sentences of $\mathcal{L}^*$ for $\alpha \le \omega$. We arrange that

(a) $\Sigma_0 = T$;

(b) $\Sigma_\omega = \bigcup\{\Sigma_n \mid n \in \omega\}$;

(c) for $i < \alpha \le \omega,\ \Sigma_i \subseteq \Sigma_\alpha$;

(d) for $\alpha \le \omega,\ \Sigma_\alpha$ is consistent;

(e) $\operatorname{card}(\Sigma_{i+1} \setminus \Sigma_i) \le 3$ for $i \in \omega$;

(f) for $i \in \omega$, either $\sigma_i \in \Sigma_{i+1}$ or $\neg\sigma_i \in \Sigma_{i+1}$;

(g) if $i \in \omega$, if σ_i is $(\exists x)\,\varphi(x)$, and if $\sigma_i \in \Sigma_{i+1}$, then $\varphi(c_j) \in \Sigma_{i+1}$ for some $j \in \omega$;

(h) for $i \in \omega$, there is some $\psi(x_1, \ldots, x_n) \in \Sigma(x_1, \ldots, x_n)$ such that $\neg\psi(c_{(i)_1}, \ldots, c_{(i)_n}) \in \Sigma_{i+1}$.

Once we carry out this construction, we can finish the proof by setting $T^* = \Sigma_\omega$.

Assume that $i \in \omega$ and that we are given $\Sigma_j,\ j \le i$, violating none of (a)–(h).

If $\Sigma_i \cup \{\neg\sigma_i\}$ is consistent, then let $\Sigma'_i = \Sigma_i \cup \{\neg\sigma_i\}$. Otherwise let $\Sigma'_i = \Sigma_i \cup \{\sigma_i\}$ unless σ_i is $(\exists y)\varphi_i(y)$ for some φ, in which case let $\Sigma'_i = \Sigma_i \cup \{\sigma_i, \varphi_i(c_j)\}$, where j is minimal such that c_j does not occur in Σ_i or σ_i.

Let $z_1, \ldots, z_n$ be distinct variables not occurring in any member of $\Sigma'_i \setminus T$. Let τ be the conjunction of all sentences in $\Sigma'_i \setminus T$. Let $\bar\tau$ come from τ by replacing, for $1 \le m \le n$, all occurrences of x_m by occurrences of z_m. (The point of this replacement is to make sure that no $c_{(i)_m}$ occurs in the scope of $(\exists x_m)$.) Let $c_{j_1}, \ldots, c_{j_k}$ be all the new constants occurring in $\bar\tau$ that are not among $c_{(i)_1}, \ldots, c_{(i)_n}$. Let $y_1, \ldots, y_k$ be variables not occurring in $\bar\tau$ that are distinct from one another and from $x_1, \ldots, x_n$. There is a formula $\chi(y_1, \ldots y_k, x_1, \ldots, x_n)$ of $\mathcal{L}$, such that $\bar\tau$ is $\chi(c_{j_1}, \ldots, c_{j_k}, c_{(i)_1}, \ldots, c_{(i)_n})$.

We show that there is a $\psi(x_1, \ldots, x_n) \in \Sigma(x_1, \ldots, x_n)$ such that $\Sigma'_i \not\models \psi(c_{(i)_1}, \ldots, c_{(i)_n})$. Assume otherwise. Then

$$T \cup \{\chi(c_{j_1}, \ldots, c_{j_k}, c_{(i)_1}, \ldots, c_{(i)_n})\} \models \psi(c_{(i)_1}, \ldots, c_{(i)_n})$$

for each $\psi(x_1, \ldots, x_n) \in \Sigma(x_1, \ldots, x_n)$. Properties (XII) and (X) of $\models$ imply that

$$T \models (\exists y_1) \cdots (\exists y_k)\, \chi(y_1, \ldots, y_k, c_{(i)_1}, \ldots, c_{(i)_n}) \rightarrow \psi(c_{(i)_1}, \ldots, c_{(i)_n})$$

for each $\psi(x_1, \ldots, x_n) \in \Sigma(x_1, \ldots, x_n)$. Let $\chi'(y_1, \ldots, y_k, x_1, \ldots, x_n)$ be the result of adding conjuncts $x_m = x_{m'}$ to $\chi(y_1, \ldots, y_k, x_1, \ldots, x_n)$ for $1 \leq m < m' \leq n$ such that $(i)_m = (i)_{m'}$. Then

$$T \models (\forall x_1) \cdots (\forall x_n)((\exists y_1) \cdots (\exists y_k)\, \chi'(y_1, \ldots, y_k, x_1, \ldots, x_n) \rightarrow \psi(x_1, \ldots, x_n))$$

for each $\psi(x_1, \ldots, x_n) \in \Sigma(x_1, \ldots, x_n)$. The sentence

$$(\exists y_1) \cdots (\exists y_k)\, \chi'(y_1, \ldots, y_k, c_{(i)_1}, \ldots, c_{(i)_n})$$

is logically implied by $\bar{\tau}$, so by τ, and so by Σ'_i. Hence this sentence is consistent with T. If the formula

$$(\exists y_1) \cdots (\exists y_k)\, \chi'(y_1, \ldots, y_k, x_1, \ldots, x_n)$$

does not belong to $\Sigma(x_1, \ldots, x_n)$, then we get a contradiction by taking its negation for $\psi(x_1, \ldots, x_n)$. Otherwise the formula witnesses that $\Sigma(x_1, \ldots, x_n)$ is principal, a contradiction.

Let $\Sigma_{i+1} = \Sigma'_i \cup \{\neg\psi(c_{(i)_1}, \ldots, c_{(i)_n})\}$ for some $\psi(x_1, \ldots, x_n)$ of the sort we have just proved to exist. □

If T is a theory and κ is a cardinal number, then T is *κ-categorical* if any two models of T of cardinal κ are isomorphic.

Theorem 3.17. *Let T be a complete theory. Then T is $\aleph_0$-categorical if and only if, for every $n \in \omega$, T has only finitely many n-types.*

Proof. Suppose first that T has, for each n, only finitely many n-types. We show that every model of T is ω-saturated, and so that every countable model of T is saturated. By Theorem 3.13, this implies that T is $\aleph_0$-categorical.

Let $\mathfrak{A}$ be a model of T. Let $Y = \{b_1, \ldots, b_m\}$ be a finite subset of A. Let $\Sigma(x_1, \ldots, x_n)$ be an n-type of $\mathrm{Th}(\mathfrak{A}_Y)$. Let $\tilde{\Sigma}(x_1, \ldots, x_{n+m})$ be

$$\{\varphi(x_1, \ldots, x_{n+m}) \mid \varphi(x_1, \ldots, x_n, c^{b_1}, \ldots, c^{b_m}) \in \Sigma(x_1, \ldots, x_n)\}\,.$$

Clearly $\tilde{\Sigma}(x_1, \ldots, x_{n+m})$ is an $(n+m)$-type of T. By Theorem 3.15, let $\tilde{\Delta}(x_1, \ldots, x_{n+m})$ witness that $\tilde{\Sigma}(x_1, \ldots, x_{n+m})$ is principal. If $c_1, \ldots, c_n$ are new constants, then

$$\{\varphi(c_1, \ldots, c_n, c^{b_1}, \ldots, c^{b_m}) \mid \varphi(x_1, \ldots, x_{n+m}) \in \tilde{\Delta}(x_1, \ldots, x_{n+m})\}$$

is consistent with $\mathrm{Th}(\mathfrak{A}_Y)$. Let $\Delta(x_1, \ldots, x_n)$ be

$$\{\varphi(x_1, \ldots, x_n, c^{b_1}, \ldots, c^{b_m}) \mid \varphi(x_1, \ldots, x_{n+m}) \in \tilde{\Delta}(x_1, \ldots, x_{n+m})\}\,.$$

Then

$$(\exists x_1) \cdots (\exists x_n) \bigwedge\!\!\!\bigwedge \Delta(x_1, \ldots, x_n) \in \mathrm{Th}(\mathfrak{A}_Y)\,.$$

Let $a_1, \ldots, a_n$ be such that $\mathfrak{A}_Y \models (\bigwedge\!\!\!\bigwedge \Delta(x_1, \ldots, x_n))[a_1, \ldots, a_n]$. Because the set $\tilde{\Delta}(x_1, \ldots, x_{n+m})$ witnesses that $\tilde{\Sigma}(x_1, \ldots, x_{n+m})$ is principal, we have that $\mathfrak{A} \models \varphi[a_1, \ldots, a_n, b_1, \ldots, b_m]$ for all $\varphi(x_1, \ldots, x_{n+m}) \in \tilde{\Sigma}(x_1, \ldots, x_{n+m})$. It follows that $\mathfrak{A}_Y \models \psi[a_1, \ldots, a_n]$ for all $\psi(x_1, \ldots, x_n) \in \Sigma(x_1, \ldots, x_n)$.

Now suppose that $n \in \omega$ and that T has infinitely many n-types. By Theorem 3.15, let $\Sigma(x_1, \ldots, x_n)$ be a non-principal n-type of T. Clearly T has no finite models. Thus it is enough to show that T has a countable model that realizes $\Sigma(x_1, \ldots, x_n)$ and a countable model that omits $\Sigma(x_1, \ldots, x_n)$. The former can be proved by a simple compactness argument and the Löwenheim–Skolem theorem. The latter is a consequence of Theorem 3.16. □

Example of an $\aleph_0$-categorical theory: Let $\mathcal{L}$ be $\{<\}$. Let T be the theory of dense linear orderings without endpoints.

To see that T is $\aleph_0$-categorical, let $\mathfrak{A}$ and $\mathfrak{B}$ be countable models of T. Let $a_1, \ldots, a_n$ be elements of A and let $b_1, \ldots, b_n$ be elements of B. Suppose suppose that $a_i \mapsto b_i$ is order preserving (i.e., that $a_i <_{\mathfrak{A}} a_j$ if and only if $b_i <_{\mathfrak{B}} b_j$). Let $a \in A$. One of the following must hold:

(i) $a = a_i$ for some i;

(ii) $a <_{\mathfrak{A}} a_i$ for all i;

(iii) $a_i <_{\mathfrak{A}} a$ for all i;

(iv) $a_i <_{\mathfrak{A}} a <_{\mathfrak{A}} a_j$ for some i and j such that there is no a_k with $a_i <_{\mathfrak{A}} a_k <_{\mathfrak{A}} a_j$.

Since $\mathfrak{B} \models T$, there must be a $b \in B$ such that whichever of (i)–(iv) holds continues to hold when each occurrence of the letter "a" is replaced by "b" and each occurrence of "$\mathfrak{A}$" is replaced by "$\mathfrak{B}$". Choosing such a b, we can send a to b and extend the given order preserving correspondence. This argument and its dual allow us to show $\mathfrak{A} \cong \mathfrak{B}$ by a construction like that of the proof of Theorem 3.13,

The argument allows us, moreover, to construct an isomorphism extending any given order preserving correspondence between finite subsets of A and B. Hence our given $\langle a_i \mid i < n\rangle$ and $\langle b_i \mid i < n\rangle$ satisfy exactly the same formulas in their respective models. This shows that each n-type $\Sigma(x_1, \ldots, x_n)$ of T is determined by a conjunction of formulas of the forms $x_i = x_j$ and $x_i < x_j$.

We finish our study of types by discussing briefly the concept of stability. For cardinal numbers κ, a theory T is *κ-stable* if, for every model $\mathfrak{A}$ of T and every $Y \subseteq A$, if $\mathrm{card}\,(Y) \leq \kappa$ then $\mathrm{Th}(\mathfrak{A}_Y)$ has $\leq \kappa$ one-types. A theory T is *stable* if T is κ-stable for some infinite κ.

Theorem 3.18. *Let T be a theory in $\mathcal{L}$. If T is ω-stable, then T is κ-stable for every infinite κ.*

Proof. Let κ be an infinite cardinal, and suppose that T is not κ-stable. Let $\mathfrak{A}_Y$ witness this fact. By recursion on $\ell\mathrm{h}(s)$, we define for each $s \in {}^{<\omega}2$, a formula $\varphi_s(x)$ of $\mathcal{L}_Y$. We shall arrange that

(a) for each $s \in {}^{<\omega}2$, $\varphi_{s\cup\{\langle \ell\mathrm{h}(s),0\rangle\}}$ is the negation of $\varphi_{s\cup\{\langle \ell\mathrm{h}(s),1\rangle\}}$;

(b) for each $s \in {}^{<\omega}2$, there are more than κ one-types of $\mathrm{Th}(\mathfrak{A}_Y)$ that include $\{\varphi_t \mid t \subseteq s\}$.

It will follow that the $\{\varphi_s \mid s \subseteq x\}$, $x \in {}^{\omega}\omega$, can be extended to distinct one types of $\mathrm{Th}(\mathfrak{A}_Z)$, where Z is the set of all $a \in Y$ such that c^a occurs in some φ_s.

Let $\varphi_\emptyset$ be $x = x$. Let $s \in {}^{<\omega}\omega$ and assume that φ_t is defined for $t \subseteq s$ and has property (b). Let $n = \ell\mathrm{h}(s)$. If we cannot define $\varphi_{s\cup\{\langle n,0\rangle\}}$ and $\varphi_{s\cup\{\langle n,1\rangle\}}$ so as to satisfy (a) and (b), then there is a type $\Sigma(x)$ of $\mathrm{Th}(\mathfrak{A}_Y)$ such that $\{\varphi_t \mid t \subseteq s\} \subseteq \Sigma(x)$ and such that, for any $\psi(x) \in \Sigma(x)$, no more than κ one-types of $\mathrm{Th}(\mathfrak{A}_Y)$ include $\{\varphi_t \mid t \subseteq s\} \cup \{\neg\psi\}$. This is a contradiction.

$\square$

Exercise 3.8. Show that the theory of dense linear orderings without endpoints is not ω-stable.

Hint. Consider the model $(\mathrm{Q}; <)$, where Q is the set of all rational numbers. Let $Y = \mathrm{Q}$.

Exercise 3.9. Let T be a complete theory with a countably saturated model. Prove that T has a model that is *atomic*, i.e., realizes no non-principal types.

Hint. Generalize Theorem 3.16.

Exercise 3.10. Let add constants c_i, $i \in \omega$ to the language $\{<\}$. Let T be gotten from the theory of dense linear orderings without endpoints by adding the additional axioms $c_i < c_j$ for $i < j \in \omega$. Prove that there are exactly 3 non-isomorphic expansions of $(\mathrm{Q}; <)$ to a model of T. Which of these is saturated and which is atomic?

Exercise 3.11. Show that the theory of algebraically closed fields of characteristic 0 is not $\aleph_0$-categorical but is κ-categorical for every uncountable cardinal κ.

We now drop our assumption that $\mathcal{L}$ is countable.

Let $\mathcal{L}$ be a language, let I be a non-empty set, and let $\mathfrak{A}_i$, $i \in I$ be models for $\mathcal{L}$. Let $\mathcal{U}$ be an ultrafilter on I.

We define $\mathfrak{A} = \prod_{i \in I} \mathfrak{A}_i / \mathcal{U}$, the *ultraproduct* of $\langle \mathfrak{A}_i \mid i \in I \rangle$ with respect to $\mathcal{U}$, as follows:

Let $\prod_{i \in I} A_i$ be the set of all functions f such that domain $(f) = I$ and each $f(i) \in A_i$. For elements f and g of $\prod_{i \in I} A_i$, define

$$f \sim_{\mathcal{U}} g \leftrightarrow \{i \in I \mid f(i) = g(i)\} \in \mathcal{U}\,.$$

Let $[\![f]\!]_{\mathcal{U}}$ be the equivalence class of f with respect to the equivalence relation $\sim_{\mathcal{U}}$. Let

$$A = \{[\![f]\!]_{\mathcal{U}} \mid f \in \prod_{i \in I} A_i\}\,.$$

If P is a k-place relation symbol of $\mathcal{L}$, let

$$P_{\mathfrak{A}}([\![f_1]\!]_{\mathcal{U}}, \dots, [\![f_n]\!]_{\mathcal{U}}) \leftrightarrow \{i \in I \mid P_{\mathfrak{A}_i}(f_1(i), \dots, f_n(i))\} \in \mathcal{U}\,.$$

If c is a constant of $\mathcal{L}$, let $c_{\mathfrak{A}} = [\![f]\!]_{\mathcal{U}}$, where $f(i) = c_{\mathfrak{A}_i}$. If F is a k-place function symbol of $\mathcal{L}$, set $F_{\mathfrak{A}}([\![f_1]\!]_{\mathcal{U}}, \dots, [\![f_n]\!]_{\mathcal{U}}) = [\![f]\!]_{\mathcal{U}}$, where $f(i) = F_{\mathfrak{A}_i}(f_1(i), \dots f_n(i))$. It is easy to check that $\mathfrak{A}$ is well-defined.

Theorem 3.19 (Łoś). (Uses Choice) *For each formula* $\varphi(x_1, \ldots, x_n)$ *of* $\mathcal{L}$ *and for any elements* $f_1, \ldots, f_n$ *of* $\prod_{i \in I} A_i$,

$$\prod_{i \in I} \mathfrak{A}_i/\mathcal{U} \models \varphi[[\![f_1]\!]_{\mathcal{U}}, \ldots, [\![f_n]\!]_{\mathcal{U}}] \leftrightarrow \{i \in I \mid \mathfrak{A}_i \models \varphi[f_1(i) \ldots, f_n(i)]\} \in \mathcal{U}\,.$$

Proof. Let $\mathfrak{A} = \prod_{i \in I} \mathfrak{A}_i/\mathcal{U}$. We proceed by induction on φ. We omit the subscipt "$\mathcal{U}$."

The case that φ is atomic is essentially by definition. If φ is $\neg\psi$, then $\mathfrak{A} \models \varphi[[\![f_1]\!], \ldots, [\![f_n]\!]]$ if and only if $\mathfrak{A} \not\models \psi[[\![f_1]\!], \ldots, [\![f_n]\!]]$. By induction, this holds if and only if $\{i \mid \mathfrak{A}_i \models \psi[f_1(i), \ldots, f_n(i)]\} \notin \mathcal{U}$. Since $\mathcal{U}$ is an ultrafilter, this holds if and only if $\{i \mid \mathfrak{A}_i \models \varphi[f_1(i), \ldots, f_n(i)]\} \in \mathcal{U}$. We omit the routine case that φ is a conjunction.

Suppose that $\varphi(x_1, \ldots, x_n)$ is $(\exists y)\, \psi(y, x_1, \ldots, x_n)$. Then

$$\begin{aligned}
\mathfrak{A} \models \varphi[[\![f_1]\!], \ldots, [\![f_n]\!]] &\leftrightarrow (\exists a \in A)\, \mathfrak{A} \models \psi[a, [\![f_1]\!], \ldots, [\![f_n]\!]] \\
&\leftrightarrow (\exists g \in \prod_{i \in I} A_i)\, \mathfrak{A} \models \psi[[\![g]\!], [\![f_1]\!], \ldots, [\![f_n]\!]] \\
&\leftrightarrow (\exists g \in \prod_{i \in I} A_i)\, \{i \mid \mathfrak{A}_i \models \psi[g(i), f_1(i), \ldots, f_n(i)]\} \in \mathcal{U} \\
&\leftrightarrow \{i \mid (\exists b \in A_i)\, \mathfrak{A}_i \models \psi[b, f_1(i), \ldots, f_n(i)]\} \in \mathcal{U} \\
&\leftrightarrow \{i \mid \mathfrak{A}_i \models \varphi[f_1(i), \ldots, f_n(i)]\} \in \mathcal{U}\,.
\end{aligned}$$

The Axiom of Choice is needed to show that the next-to-last line implies the third-to-last line. $\square$

Exercise 3.12. Use ultraproducts to prove the Compactness Theorem.

Hint. Let Σ be a set of sentences every finite subset of which is consistent. Let I be the set of all finite subsets of Σ.

Exercise 3.13. Assume that $\mathcal{L}$ is countable. Let $\mathcal{U}$ be a *non-principal* ultrafilter on a countable set I, i.e., an ultrafilter to which no singleton $\{i\}$ belongs. Prove that every ultraproduct $\prod_{i \in I} \mathfrak{A}_i/\mathcal{U}$ is $\aleph_1$-saturated.

Hint. First show that there are elements U_k, $k \in \omega$, of $\mathcal{U}$ such that $\bigcap_{k \in \omega} U_k \notin \mathcal{U}$ (indeed, so that the intersection is empty). Then, given a one-type, choose f so that, for $i \in \bigcap_{k' < k} U_{k'} \setminus U_k$, $f(i)$ satisfies in $\mathfrak{A}_i$ the first k formulas of the type.

Exercise 3.14. An ultraproduct of the form $\prod_{i \in I} \mathfrak{A}/\mathcal{U}$ is called an *ultrapower*, and we may call it $\mathfrak{A}^I/\mathcal{U}$. For any ultrapower $\mathfrak{A}^I/\mathcal{U}$, define an elementary embedding $j : \mathfrak{A} \prec \mathfrak{A}^I/\mathcal{U}$.

Exercise 3.15. For κ a cardinal number, an ultrafilter $\mathcal{U}$ is *κ-complete* if the intersection of any set of fewer than κ elements of $\mathcal{U}$ belongs to $\mathcal{U}$. A *countably complete* ultrafilter is one that is $\aleph_1$-complete.

Suppose that $\mathcal{U}$ is a countably complete ultrafilter on a set I. Let A be any set. Let $\mathfrak{B} = (A; \in \restriction A)^I/\mathcal{U}$.

(a) Prove that $\in_{\mathfrak{B}}$ is wellfounded.

(b) Suppose that $\mathcal{U}$ is non-principal. (The existence of a countably complete, non-principal ultrafilter cannot be proved in ZFC.) Show that there is a largest cardinal κ such that $\mathcal{U}$ is κ-complete. Assume that A is transitive and that $\kappa \in A$. Let j be the embedding defined in the (natural) solution to Exercise 3.14. Let B_0 be the set of all "ordinals" of $\mathfrak{B}$. Prove that $\pi : (B_0; \in_{\mathfrak{B}} \restriction B_0) \cong (\beta; \in \restriction \beta)$ for some ordinal β and some π. Prove that κ is the smallest ordinal number α such that $\pi(j(\alpha)) \neq \alpha$.

Exercise 3.16. The solution to Exercise 3.12 suggested by the hint used Choice (1) to get that every filter can be extended to an ultrafilter and (2) because the proof of Theorem 3.19 used Choice. (One use was mentioned; an implicit use was to get $\prod_{i \in I} A_i$ non-empty.) Eliminate the uses (2) of Choice by employing a different I from that suggested in the hint to Exercise 3.12.

In the next section we shall study theories and models of arithmetic. We close the section on model theory by proving a result about a fragment of the main theory of the next section.

If $i_1 < \cdots < i_n$ and if $\varphi(v_{i_1}, \ldots, v_{i_n})$ is a formula containing free occurrences of all of $v_{i_1}, \ldots, v_{i_n}$, then the *universal closure* of φ is the sentence $(\forall v_{i_1}) \cdots (\forall v_{i_n})\, \varphi$. The *universal closure* of a sentence is the sentence itself.

If Σ is a set of sentences and φ is a formula, then we say $\Sigma \models \varphi$ if $\Sigma \models$ the universal closure of φ. A formula is *valid* if its universal closure is valid. Formulas φ and ψ are *equivalent* if $(\varphi \leftrightarrow \psi)$ is valid, and φ and ψ are *equivalent in* T, for T a theory, if $T \models (\varphi \leftrightarrow \psi)$.

A theory T in a language $\mathcal{L}$ *admits elimination of quantifiers* if every formula of $\mathcal{L}$ is equivalent in T to a quantifier-free formula.

Theorem 3.20. *Let T be a theory. Assume that, for every formula φ of the form*

$$(\exists x)(\chi_1 \wedge \ldots \wedge \chi_n),$$

with each χ_j atomic or the negation of an atomic formula, is equivalent in T to a quantifier-free formula. Then T admits elimination of quantifiers.

Proof. Suppose first that φ is of the form $(\exists x)\,\psi$, with ψ quantifier-free. It is easy to see that every quantifier-free formula is equivalent to one of the form

$$((\chi_{1,1} \wedge \ldots \wedge \chi_{1,n_1}) \vee \ldots \vee (\chi_{m,1} \wedge \ldots \wedge \chi_{m,n_m}))\,,$$

with each $\chi_{i,j}$ atomic or negation of atomic. Thus we assume φ is

$$(\exists x)((\chi_{1,1} \wedge \ldots \wedge \chi_{1,n_1}) \vee \ldots \vee (\chi_{m,1} \wedge \ldots \wedge \chi_{m,n_m}))\,,$$

with each $\chi_{i,j}$ atomic or negation of atomic. But this formula is equivalent to

$$((\exists x)(\chi_{1,1} \wedge \ldots \wedge \chi_{1,n_1}) \vee \ldots \vee (\exists x)(\chi_{m,1} \wedge \ldots \wedge \chi_{m,n_m}))\,.$$

By hypothesis, each of the disjuncts is equivalent in T to a quantifier-free formula. Hence φ is equivalent in T to a quantifier free-formula.

The theorem now follows easily by induction on φ. $\square$

Let $\mathcal{L} = \{\mathbf{0}, \mathbf{S}\}$, where $\mathbf{0}$ is a constant and $\mathbf{S}$ is a one-place function symbol. For $n \in \omega$, let us abbreviate

$$\underbrace{\mathbf{S}(\cdots\mathbf{S}(}_{n}\,t\,\underbrace{)\cdots)}_{n}$$

by $\mathbf{S}^n(t)$.

Theorem 3.21. $\mathrm{Th}(\omega; 0, \mathcal{S})$ *admits elimination of quantifiers.*

Proof. Let $T = \mathrm{Th}(\omega; 0, \mathcal{S})$. By Theorem 3.20, it suffices to prove that every formula of the form $(\exists x)(\chi_1 \wedge \ldots \wedge \chi_n)$, with each χ_i atomic or negation of atomic, is equivalent in T to an quantifier-free formula. Fix a formula of this form.

If ψ and ψ' are formulas and ψ does not contain a free occurrence of the variable y, then $(\exists y)(\psi \wedge \psi')$ is equivalent to $\psi \wedge (\exists y)\psi'$. Thus we may assume that each χ_i has an occurrence of the variable x.

By the symmetry of identity, each atomic formula of $\mathcal{L}$ that contains an occurrence of x is equivalent to one of the form

$$\mathbf{S}^j(x) = \mathbf{S}^k(t)$$

where t is $\mathbf{0}$ or a variable. If t is x, then $\mathbf{S}^j(x) = \mathbf{S}^k(t)$ is equivalent in T to $\mathbf{0} = \mathbf{0}$ if $j = k$ and equivalent to $\mathbf{0} \neq \mathbf{0}$ if $j \neq k$. Thus we may assume that for each χ_i there are j_i and t_i, where t_i is a term not containing x, such that χ_i is the equation $\mathbf{S}^{j_i}(x) = t_i$ or else is the negation of this equation.

If each χ_i is the negation of an equation, then it is evident that $(\omega; 0, \mathcal{S})$ satisfies the universal closure of $(\exists x)(\chi_1 \wedge \ldots \wedge \chi_n)$, so this formula is equivalent in T to $\mathbf{0} = \mathbf{0}$.

Suppose then that some χ_i is $\mathbf{S}^{j_i}(x) = t_i$. In each χ_m, $m \neq i$, we replace $\mathbf{S}^{j_m}(x) = t_m$ by $\mathbf{S}^{j_m}(t_i) = \mathbf{S}^{j_i}(t_m)$. If we also replace χ_i by

$$t_i \neq \mathbf{0} \wedge \ldots \wedge t_i \neq \mathbf{S}^{j_i - 1}(\mathbf{0})$$

(or by $\mathbf{0} = \mathbf{0}$ if $j_i = 0$), then we get a formula equivalent in T to our original one. The new formula is $(\exists x)\,\psi$, where ψ has no occurrences of x; so it is equivalent to the quantifier-free formula ψ. $\square$

From the proof just given, one can extract a list of axioms for $\mathrm{Th}(\omega; 0, \mathcal{S})$ (an infinite list). This gives us a *decision procedure* for $\mathrm{Th}(\omega; 0, \mathcal{S})$, an algorithm for deciding whether any given sentence belongs to the theory. (Simply list all deductions from the axioms until one is found of the sentence or its negation.) The proofs of Theorems 3.20 and 3.21 also directly provide a decision procedure.

4 Incompleteness

The theory of $(\omega; 0, \mathcal{S}, <)$, like that of $(\omega; \mathcal{S})$, admits elimination of quantifiers and is decidable (has a decision procedure). The same is essentially true of $\mathrm{Th}(\omega; 0, \mathcal{S}, <, +)$, though this theory doesn't literally admit elimination of quantifiers.

Let $\mathfrak{N} = (\omega; 0, \mathcal{S}, <, +, \cdot)$. The situation with $\mathrm{Th}(\mathfrak{N})$ is quite different from that of its reducts just mentioned. We shall see in this section just how different it is.

Let $\mathcal{L}^{\mathrm{P}A}$ be the language $\{\mathbf{0}, \mathbf{S}, <, +, \cdot\}$, for which we take $\mathfrak{N}$ to be a model.

Peano Arithmetic (PA) is the natural attempt to axiomatize $\mathfrak{N}$. Peano Arithmetic is the set of sentences implied by the following axioms:

Axioms for PA.

(a) Universal closures of the following formulas (where we employ some obvious abbreviations and conventions):

(1) $\mathbf{0} \neq \mathbf{S}(v_0)$;

(2) $\mathbf{S}(v_0) = \mathbf{S}(v_1) \to v_0 = v_1$;

(3) $v_0 \not< \mathbf{0}$;

(4) $v_0 < \mathbf{S}(v_1) \leftrightarrow v_0 \leq v_1$;

(5) $v_0 + \mathbf{0} = v_0$;

(6) $v_0 + \mathbf{S}(v_1) = \mathbf{S}(v_0 + v_1)$;

(7) $v_0 \cdot \mathbf{0} = \mathbf{0}$;

(8) $v_0 \cdot \mathbf{S}(v_1) = (v_0 \cdot v_1) + v_0$.

(b) The *Schema of Induction*, consisting of the universal closures of all formulas of the form:

$$((\varphi(\mathbf{0}, x_1, \ldots, x_n) \wedge (\forall x_0)(\varphi(x_0, \ldots x_n) \to \varphi(\mathbf{S}(x_0), x_1, \ldots, x_n))) \\ \to (\forall x_0)\, \varphi(x_0, \ldots, x_n)) \, .$$

We wish to study a sufficiently strong *finitely axiomatizable* subtheory of PA. For technical reasons, it is easier to work in a language with exponentiation, so we first consider a theory QE which is not literally a subtheory of PA. (In this, and in some other things, we are following Herbert Enderton's *A Mathematical Introduction to Logic*.)

Let $\mathcal{L}^{\mathrm{PAE}} = \mathcal{L}^{\mathrm{PA}} \cup \{\mathbf{E}\}$.

QE is the set of sentences implied by Axioms (1)–(8) above and two additional axioms, the universal closures of:

(9) $v_0 \, \mathbf{E} \, \mathbf{0} = \mathbf{S}(\mathbf{0})$;

(10) $v_0 \, \mathbf{E} \, \mathbf{S}(v_1) = (v_0 \, \mathbf{E} \, v_1) \cdot v_0$.

Let $\mathfrak{N}' = (\omega; 0, \mathcal{S}, <, +, \cdot, E)$, where $E(a,b) = a^b$. Clearly $\mathfrak{N}' \models \mathrm{QE}$.

Lemma 4.1. *For all $k \in \omega$,*

$$\mathrm{QE} \models (x < \mathbf{S}^{k+1}(\mathbf{0}) \leftrightarrow (x = \mathbf{0} \vee \ldots \vee x = \mathbf{S}^k(\mathbf{0}))) .$$

Proof. We proceed by induction on k. By Axiom (4),

$$\mathrm{QE} \models (x < \mathbf{S}^{k+1}(\mathbf{0}) \leftrightarrow (x < \mathbf{S}^k(\mathbf{0}) \vee x = \mathbf{S}^k(\mathbf{0}))) .$$

If $k = 0$, our conlusion follows by Axiom (3). If $k > 0$, it follows by induction. □

Lemma 4.2. *If t is a term without variables and $k = t_{\mathfrak{N}'}$, then*

$$\mathrm{QE} \models t = \mathbf{S}^k(\mathbf{0}) .$$

Proof. We use induction on the complexity of t. The case that t is $\mathbf{0}$ is immediate.

Assume that t is $\mathbf{S}(u)$. By induction, $\mathrm{QE} \models u = \mathbf{S}^{u_{\mathfrak{N}'}}(\mathbf{0})$. Hence $\mathrm{QE} \models \mathbf{S}(u) = \mathbf{S}^{u_{\mathfrak{N}'}+1}(\mathbf{0})$.

Assume next that t is $u_1 + u_2$. Let $j_1 = (u_1)_{\mathfrak{N}'}$ and let $j_2 = (u_2)_{\mathfrak{N}'}$. By induction, $\mathrm{QE} \models u_1 = \mathbf{S}^{j_1}(\mathbf{0})$ and $\mathrm{QE} \models u_2 = \mathbf{S}^{j_2}(\mathbf{0})$. Axiom (5) and j_2 applications of Axiom (6) give that

$$\mathrm{QE} \models \mathbf{S}^{j_1}(\mathbf{0}) \mathbf{+} \mathbf{S}^{j_2}(\mathbf{0}) = \mathbf{S}^{j_1+j_2}(\mathbf{0}) .$$

Applications of Axioms (7) and (8) give that $\mathrm{QE} \models \mathbf{S}^{j_1}(\mathbf{0}) \cdot \mathbf{S}^{j_2}(\mathbf{0}) = \mathbf{S}^{j_1 \cdot j_2}(\mathbf{0})$, for any j_1 and $j_2 \in \omega$. This allows us to handle the case that t is $u_1 \cdot u_2$. The case that t is $u_1 \, \mathbf{E} \, u_2$ is treated similarly, using Axioms (9) and (10). □

Let T be a theory in a language $\mathcal{L}$ containing $\mathbf{0}$ and $\mathbf{S}$. A formula $\varphi(v_1, \ldots, v_n)$ of $\mathcal{L}$ *represents* $R \subseteq {}^n\omega$ *in* T if, for all elements $a_1, \ldots, a_n$ of ω,

$$\begin{aligned} R(a_1, \ldots, a_n) &\rightarrow T \models \varphi(\mathbf{S}^{a_1}(\mathbf{0}), \ldots, \mathbf{S}^{a_n}(\mathbf{0})) ; \\ \neg R(a_1, \ldots, a_n) &\rightarrow T \models \neg\varphi(\mathbf{S}^{a_1}(\mathbf{0}), \ldots, \mathbf{S}^{a_n}(\mathbf{0})) . \end{aligned}$$

If some formula representents R in T, then we say that R is *representable* in T.

Representability is related to definability. If $\mathfrak{A}$ is a model and $R \subseteq {}^nA$, then R is *definable in* $\mathfrak{A}$ if there is a formula $\varphi(v_1, \ldots, v_n)$ of $\mathcal{L}$ such that, for any members $a_1, \ldots, a_n$ of A,

$$R(a_1, \ldots, a_n) \leftrightarrow \mathfrak{A} \models \varphi[a_1, \ldots, a_n] .$$

For such a φ, we say that φ *defines* R *in* $\mathfrak{A}$. The relation between representability and definability is the following. Suppose that $\mathfrak{A}$ is a model of a theory T in a language containing $\mathbf{0}$ and $\mathbf{S}$. Suppose also that $A = \omega$, that $\mathbf{0}_{\mathfrak{A}} = 0$, and that $\mathbf{S}_{\mathfrak{A}} = \mathcal{S}$. Then any formula that represents a relation in T also defines that relation in $\mathfrak{A}$. The converse is not in general true.

We shall define representability of functions as well as of relations. A natural definition would be: "$\varphi(v_1, \ldots, v_{n+1})$ represents f in T if and only if φ represents the graph of f in T," where the *graph of* f is the $(n+1)$-ary relation that holds of $(a_1, \ldots, a_{n+1})$ if and only if $f(a_1, \ldots, a_n) = a_{n+1}$. For technical reasons, we shall define a stronger notion, though it will turn out that the two notions are equivalent for any T containing Axioms (1)–(4).

If $f : {}^n\omega \to \omega$ and T is a theory in a language containing $\mathbf{0}$ and $\mathbf{S}$, then a formula $\varphi(v_1, \ldots, v_{n+1})$ *represents* f *in* T if, for all $a_1, \ldots, a_n$,

$$T \models (\forall v_{n+1})(\varphi(\mathbf{S}^{a_1}(\mathbf{0}), \ldots, \mathbf{S}^{a_n}(\mathbf{0}), v_{n+1}) \leftrightarrow v_{n+1} = \mathbf{S}^{f(a_1, \ldots, a_n)}(\mathbf{0})) .$$

Say that f is *representable in* T if some formula represents f in T.

Note that if T contains Axioms (1) and (2) and φ represents f in T then φ represents the graph of f in T. We shall say that T *proves* $\varphi(v_1, \ldots, v_{n+1})$ *functional* if

$$T \models (\forall v_1) \cdots (\forall v_n)(\exists v_{n+1})(\forall v_{n+2})(\varphi(v_1, \ldots, v_n, v_{n+2}) \leftrightarrow v_{n+2} = v_{n+1}) .$$

If T proves $\varphi(v_1, \ldots, v_{n+1})$ functional and φ represents the graph of f in T, then φ represents f in T. The converse does not hold in general.

Exercise 4.1. Show that, for every sentence σ of $\mathcal{L}^{\mathrm{PAE}}$ that is atomic or negation of atomic,

$$\mathrm{QE} \models \sigma \leftrightarrow \mathfrak{N}' \models \sigma .$$

Exercise 4.2. A formula φ of $\mathcal{L}^{\mathrm{PAE}}$ belongs to Δ_0 (or, as we shall say, *is* Δ_0) if φ belongs to the smallest set containing the atomic formulas and closed under negation, conjunction and *bounded quantification.* Closure of Δ_0 under bounded quantification means that

$$\psi \in \Delta_0 \rightarrow \begin{cases} (\exists x)(x{<}t \ \wedge \ \psi) \in \Delta_0\,; \\ (\exists x)(x{\leq}t \ \wedge \ \psi) \in \Delta_0\,, \end{cases}$$

for any term t not containing x. The Σ_1 formulas of $\mathcal{L}^{\mathrm{PAE}}$ are those of the form $(\exists x_1)\cdots(\exists x_n)\,\psi$, where ψ is Δ_0.

(a) Prove that, for any Δ_0 sentence σ, $\mathrm{QE} \models \sigma \ \leftrightarrow \ \mathfrak{N}' \models \sigma$.

(b) Prove that, for any Σ_1 sentence σ, $\mathrm{QE} \models \sigma \ \leftrightarrow \ \mathfrak{N}' \models \sigma$.

A function is *primitive recursive* just in case (I)-(III) below require it to be. (I.e., the primitive recursive functions form the smallest set of functions containing the functions of (I) and closed under the operations of (II) and (III).)

(I) The following are primitive recursive.

(a) $\mathcal{S} : \omega \rightarrow \omega\,;$

(b) $I^n_i : {}^n\omega \rightarrow \omega$, for $1 \leq i \leq n \in \omega$, where $I^n_i(a_1, \ldots, a_n) = a_i\,;$

(c) All constant functions $f : {}^n\omega \rightarrow \omega\,.$

(II) (Composition) If $f : {}^m\omega \rightarrow \omega$ and $g_1, \ldots, g_m : {}^n\omega \rightarrow \omega$ are primitive recursive, then so is h, where

$$h(a_1, \ldots, a_n) = f(g_1(a_1, \ldots, a_n), \ldots, g_m(a_1, \ldots, a_n))\,.$$

(III) (Primitive Recursion) If $f : {}^n\omega \rightarrow \omega$ and $g : {}^{n+2}\omega \rightarrow \omega$ are primitive recursive, then so is h, where

$$\begin{aligned} h(a_1, \ldots, a_n, 0) &= f(a_1, \ldots, a_n)\,; \\ h(a_1, \ldots, a_n, \mathcal{S}(b)) &= g(a_1, \ldots, a_n, b, h(a_1, \ldots, a_n, b))\,. \end{aligned}$$

We allow functions of zero arguments (e.g., the f of (III)), all of which are primitive recursive by (I)(c).

A function is *recursive* just in case it is required to be by (I)-(III), with “primitive recursive” replaced by “recursive,” plus (IV) below.

(IV) (μ-Operator) If $g : {}^{n+1}\omega \rightarrow \omega$ is recursive and

$$(\forall a_1 \in \omega)\cdots(\forall a_n \in \omega)(\exists b \in \omega)\, g(a_1, \ldots, a_n, b) = 0\,,$$

then f is recursive, where

$$f(a_1, \ldots, a_n) = \mu b\, g(a_1, \ldots, a_n, b) = 0\,,$$

and where “μb” means “the least b.”

Lemma 4.3. *The relations and functions representable in* QE *are closed under complement, intersection, union, and bounded quantification. Intersection and union we construe as operations acting on pairs of relations that are subsets of the same* ${}^n\omega$*. Bounded quantification is the operation* $\langle f, R\rangle \mapsto R'$*, where*

$$R'(a_1,\ldots,a_n) \leftrightarrow (\exists a_{n+1})(a_{n+1} < f(a_1,\ldots,a_n) \wedge R(a_1,\ldots,a_{n+1}))\,.$$

Proof. If φ represents R, then $\neg\varphi$ represents the complement of R; if φ and ψ represent R and R' respectively, then $\varphi \wedge \psi$ represents $R \cap R'$; if φ and ψ represent R and R' respectively, then $\varphi \vee \psi$ represents $R \cup R'$.

To prove closure under bounded quantification, assume that $\varphi(v_1,\ldots,v_{n+1})$ and $\psi(v_1,\ldots,v_{n+1})$ represent f and R respectively.

Let $\chi(v_1,\ldots,v_n)$ be, for some appropriate variable z,

$$(\exists v_{n+1})(\exists z)(\varphi(v_1,\ldots,v_n,z) \wedge v_{n+1}{<}z \wedge \psi(v_1,\ldots,v_n,v_{n+1}))\,.$$

To see that χ represents R' in QE, fix numbers $a_1,\ldots,a_n$. Since φ represents f, we have that

$$\mathrm{QE} \models (\forall z)(\varphi(\mathbf{S}^{a_1}(\mathbf{0}),\ldots,\mathbf{S}^{a_n}(\mathbf{0}),z) \leftrightarrow z = \mathbf{S}^{f(a_1,\ldots,a_n)}(\mathbf{0}))\,.$$

Thus $\chi(\mathbf{S}^{a_1}(\mathbf{0}),\ldots,\mathbf{S}^{a_n}(\mathbf{0}))$ is equivalent in QE to

$$(\exists v_{n+1})(v_{n+1}{<}\mathbf{S}^{f(a_1,\ldots,a_n)}(\mathbf{0}) \wedge \psi(\mathbf{S}^{a_1}(\mathbf{0}),\ldots,\mathbf{S}^{a_n}(\mathbf{0}),v_{n+1}))\,.$$

By Lemma 4.1, $\chi(\mathbf{S}^{a_1}(\mathbf{0}),\ldots,\mathbf{S}^{a_n}(\mathbf{0}))$ is equivalent in QE to

$$\psi(\mathbf{S}^{a_1}(\mathbf{0}),\ldots,\mathbf{S}^{a_n}(\mathbf{0}),\mathbf{0}) \vee \ldots \vee \psi(\mathbf{S}^{a_1}(\mathbf{0}),\ldots,\mathbf{S}^{a_n}(\mathbf{0}),\mathbf{S}^{f(a_1,\ldots,a_n)-1}(\mathbf{0}))\,,$$

(or, say, $\mathbf{0} \neq \mathbf{0}$ if $f(a_1,\ldots,a_n) = 0$). Since ψ represents R, this formula is provable or refutable in QE according to whether or not $R'(a_1,\ldots,a_n)$ holds. □

Lemma 4.4. *All the functions under clause (I) (in the definition of the primitive recursive functions) are representable in* QE*.*

Proof. Their graphs are represented by atomic formulas which QE (indeed, every theory) proves functional. □

Lemma 4.5. *The functions representable in* QE *are closed under composition (II).*

Proof. Given representable f and $g_1, \ldots, g_m$, as in the statement of (II), let $\psi_1(v_1, \ldots, v_{n+1}), \ldots, \psi_m(v_1, \ldots, v_{n+1})$ represent $g_1, \ldots, g_m$ respectively and let $\chi(v_1, \ldots, v_{m+1})$ represent f. Let $\varphi(v_1, \ldots, v_{n+1})$ be, for appropriate variables $x_1, \ldots, x_m$,

$$\begin{aligned}(\exists x_1) \cdots (\exists x_m)(\psi_1(v_1, \ldots, v_n, x_1) \wedge \ldots \\ \wedge \psi_m(v_1, \ldots, v_n, x_m) \wedge \chi(x_1, \ldots, x_m, v_{n+1}))\,.\end{aligned}$$

Let $a_1, \ldots, a_n \in \omega$. For each j,

$$\mathrm{QE} \models \psi_j(\mathbf{S}^{a_1}(\mathbf{0}), \ldots, \mathbf{S}^{a_n}(\mathbf{0}), x_j) \leftrightarrow x_j = \mathbf{S}^{g_j(a_1,\ldots,a_n)}(\mathbf{0})\,.$$

Thus QE $\models$

$$\varphi(\mathbf{S}^{a_1}(\mathbf{0}), \ldots, \mathbf{S}^{a_n}(\mathbf{0}), v_{n+1}) \leftrightarrow \chi(\mathbf{S}^{g_1(a_1,\ldots,a_n)}(\mathbf{0}), \ldots, \mathbf{S}^{g_m(a_1,\ldots a_n)}(\mathbf{0}), v_{n+1})\,.$$

But QE $\models$

$$\begin{aligned}\chi(\mathbf{S}^{g_1(a_1,\ldots,a_n)}(\mathbf{0}), \ldots, \mathbf{S}^{g_m(a_1,\ldots,a_n)}(\mathbf{0}), v_{n+1}) \\ \leftrightarrow v_{n+1} = \mathbf{S}^{f(g_1(a_1,\ldots,a_n),\ldots,g_m(a_1,\ldots a_n))}(\mathbf{0}))\,.\end{aligned}$$

Therefore QE $\models$

$$(\forall v_{n+1})(\varphi(\mathbf{S}^{a_1}(\mathbf{0}), \ldots, \mathbf{S}^{a_n}(\mathbf{0}), v_{n+1}) \leftrightarrow v_{n+1} = \mathbf{S}^{f(g_1(a_1,\ldots,a_n),\ldots,g_m(a_1,\ldots a_n))}(\mathbf{0}))\,.$$

$\square$

Lemma 4.6. *A relation R is representable in* QE *if and only if its characteristic function K_R is representable in* QE, *where*

$$K_R(a_1, \ldots, a_n) = \begin{cases} 1 & \text{if } R(a_1, \ldots, a_n)\,; \\ 0 & \text{if } \neg R(a_1, \ldots, a_n)\,. \end{cases}$$

Proof. The proof is routine, and we omit it. $\square$

Our next goal is to show that the functions representable in QE are closed under the μ-operator (IV). This would be easy if the sentence $(\forall v_1)(\forall v_2)(v_1 < v_2 \vee v_1 = v_2 \vee v_2 < v_1)$ were provable in QE. We could have made this sentence an axiom of a strengthening of QE, as does Enderton in the book cited earlier. But we did not do this, so our argument will be a little complicated.

Let $\mathbf{WC}(v_1)$ be the following formula:

$$(\mathbf{0} \leq v_1 \wedge (\forall v_2)(v_2 < v_1 \rightarrow \mathbf{S}(v_2) \leq v_1))\,.$$

Think of **WC** as "weakly comparable."

Lemma 4.7. *For every natural number k,*

(a) QE $\models$ $\mathbf{WC}(\mathbf{S}^k(\mathbf{0}))$ *;*

(b) QE $\models$ $(\forall v_1)(\mathbf{WC}(v_1) \to (v_1{<}\mathbf{S}^k(\mathbf{0}) \vee v_1 = \mathbf{S}^k(\mathbf{0}) \vee \mathbf{S}^k(\mathbf{0}){<}v_1))$.

Proof. That QE $\models \mathbf{WC}(\mathbf{0})$ follows from Axiom (3). Fix $k > 0$. By Exercise 4.1 (or by Lemma 4.1), we know that QE $\models \mathbf{0}{\leq}\mathbf{S}^k(\mathbf{0})$. An application of Lemma 4.1 gives that

$$\text{QE} \models v_2{<}\mathbf{S}^k(\mathbf{0}) \to (v_2 = \mathbf{0} \vee \ldots \vee v_2 = \mathbf{S}^{k-1}(\mathbf{0}))\,.$$

But then

$$\text{QE} \models v_2{<}\mathbf{S}^k(\mathbf{0}) \to (\mathbf{S}(v_2) = \mathbf{S}^1(\mathbf{0}) \vee \ldots \vee \mathbf{S}(v_2) = \mathbf{S}^k(\mathbf{0}))\,.$$

(a) follows by Lemma 4.1.

We prove (b) by induction on k. The case $k = 0$ comes from the first conjunct of $\mathbf{WC}(v_1)$. For the induction step note that, by Axiom (4), QE $\models$ $(v_1{\leq}\mathbf{S}^k(\mathbf{0}) \to v_1{<}\mathbf{S}(\mathbf{S}^k(\mathbf{0})))$ and that, by the second conjunct of $\mathbf{WC}(v_1)$,

$$\text{QE} \models (\mathbf{S}^k(\mathbf{0}){<}v_1 \wedge \mathbf{WC}(v_1)) \to \mathbf{S}(\mathbf{S}^k(\mathbf{0})){\leq}v_1\,. \qquad \square$$

Lemma 4.8. *The functions representable in* QE *are closed under the μ-operator (IV).*

Proof. Suppose that $\varphi(v_1, \ldots, v_{n+2})$ represents g in QE and suppose that

$$(\forall a_1 \in \omega) \cdots (\forall a_n \in \omega)(\exists b \in \omega)\, g(a_1, \ldots, a_n, b) = 0\,.$$

Let f be given by

$$f(a_1, \ldots, a_n) = \mu b\, g(a_1, \ldots, a_n, b) = 0\,.$$

Let $\psi(v_1, \ldots, v_{n+1})$ be, for an appropriate z,

$$\mathbf{WC}(v_{n+1}) \wedge \varphi(v_1, \ldots, v_{n+1}, \mathbf{0}) \wedge (\forall z)(z{<}v_{n+1} \to \neg\varphi(v_1, \ldots, v_n, z, \mathbf{0}))\,.$$

To see that ψ represents f in QE, fix $a_1, \ldots, a_n$. Using part (a) of Lemma 4.7 and the fact that φ represents g in QE, we deduce that

$$\text{QE} \models \mathbf{WC}(\mathbf{S}^{f(a_1,\ldots,a_n)}(\mathbf{0})) \wedge \varphi(\mathbf{S}^{a_1}(\mathbf{0}), \ldots, \mathbf{S}^{a_n}(\mathbf{0}), \mathbf{S}^{f(a_1,\ldots,a_n)}(\mathbf{0}), \mathbf{0}),.$$

Using the fact that φ represents g in QE and using Lemma 4.1, we get that

$$\text{QE} \models (\forall z)(z{<}\mathbf{S}^{f(a_1\ldots,a_n)}(\mathbf{0}) \to \neg\varphi(\mathbf{S}^{a_1}(\mathbf{0}), \ldots, \mathbf{S}^{a_n}(\mathbf{0}), z, \mathbf{0}))\,.$$

Combining these two facts we get that

$$\mathrm{QE} \models \psi(\mathbf{S}^{a_1}(\mathbf{0}), \ldots, \mathbf{S}^{a_n}(\mathbf{0}), \mathbf{S}^{f(a_1,\ldots,a_n)}(\mathbf{0}))\,.$$

Moreover, the second of the two facts and part (b) of Lemma 4.7 give that

$$\mathrm{QE} \models (\forall z)((\mathbf{WC}(z) \wedge \varphi(\mathbf{S}^{a_1}(\mathbf{0}), \ldots, \mathbf{S}^{a_n}(\mathbf{0}), z, \mathbf{0})) \rightarrow \mathbf{S}^{f(a_1\ldots,a_n)}(\mathbf{0}) \leq z)\,.$$

Since $\mathbf{WC}(z)$ and $\varphi(\mathbf{S}^{a_1}(\mathbf{0}), \ldots, \mathbf{S}^{a_n}(\mathbf{0}), z, \mathbf{0})$ are conjuncts of the formula $\psi(\mathbf{S}^{a_1}(\mathbf{0}), \ldots, \mathbf{S}^{a_n}(\mathbf{0}), z)$,

$$\mathrm{QE} \models (\forall z)(\psi(\mathbf{S}^{a_1}(\mathbf{0}), \ldots, \mathbf{S}^{a_n}(\mathbf{0}), z) \rightarrow \mathbf{S}^{f(a_1\ldots,a_n)}(\mathbf{0}) \leq z)\,.$$

Since $\mathrm{QE} \models \varphi(\mathbf{S}^{a_1}(\mathbf{0}), \ldots, \mathbf{S}^{a_n}(\mathbf{0}), \mathbf{S}^{f(a_1,\ldots,a_n)}(\mathbf{0}), \mathbf{0})$, consideration of the last conjunct of $\psi(\mathbf{S}^{a_1}(\mathbf{0}), \ldots, \mathbf{S}^{a_n}(\mathbf{0}), z)$ shows us that

$$\mathrm{QE} \models (\forall z)(\psi(\mathbf{S}^{a_1}(\mathbf{0}), \ldots, \mathbf{S}^{a_n}(\mathbf{0}), z) \rightarrow \mathbf{S}^{f(a_1,\ldots,a_n)}(\mathbf{0}) \not< z)\,.$$

Thus

$$\mathrm{QE} \models (\forall z)(\psi(\mathbf{S}^{a_1}(\mathbf{0}), \ldots, \mathbf{S}^{a_n}(\mathbf{0}), z) \rightarrow z = \mathbf{S}^{f(a_1,\ldots,a_n)}(\mathbf{0}))\,. \qquad \square$$

Corollary 4.9. *A function is representable in* QE *if its graph is representable in* QE.

Proof. Let R be the graph of $f : n^\omega \rightarrow \omega$.

$$f(a_1, \ldots, a_n) = \mu b\, K_{\neg R}(a_1, \ldots, a_n, b) = 0\,.$$

Lemma 4.10. *The relation $<$ and the functions $+$, $\cdot$, and E are representable in QE.*

Proof. By Exercise 4.1, $<$ and the graphs of $+$, $\cdot$, and E are represented by $v_1 {<} v_2$, $v_1 {+} v_2 = v_3$, $v_1 {\cdot} v_2 = v_3$, and $v_1\, \mathbf{E}\, v_2 = v_3$ respectively. Use Corollary 4.9 or the fact that very theory proves the last three formulas functional. $\square$

Lemma 4.11. $\{(a, b) \mid a \text{ divides } b\}$ *is representable in* QE.

Proof. For any a and b belonging to ω,

$$a \text{ divides } b \leftrightarrow (\exists c \leq b)\, a \cdot c = b\,. \qquad \square$$

Lemma 4.12. *(a) The set of all prime numbers is representable in* QE*.*

(b) The set of all pairs of adjacent primes is representable in QE*, where* (a, b) *is a pair of adjacent primes if and only if* $a < b$*, both* a *and* b *are prime, and there is no prime* c *such that* $a < c < b$*.*

Proof. The proof is an easy application of the closure of the representable functions under bounded quantification. □

Lemma 4.13. *The function* $a \mapsto p_a$ *is representable in* QE*, where* p_a *is the* $a + 1$*st prime.*

Proof. We shall show that, for any a and b belonging to ω, $p_a = b$ if and only if b is prime and there is a $c \leq b^{a^2}$ such that

(i) 2 does not divide c;

(ii) For all $q < b$ and all $r \leq b$, if (q, r) is a pair of adjacent primes, then

$$(\forall j < c)(q^j \text{ divides } c \leftrightarrow r^{j+1} \text{ divides } c)\,.$$

(iii) b^a divides c and b^{a+1} does not.

To see this, fix a and b and first note that if $p_a = b$ and

$$c = p_0^0 \cdot p_1^1 \cdot \ldots \cdot p_a^a\,,$$

then $c \leq b^{a^2}$ and c satisfies (i)–(iii).

Suppose that b is prime and that c satisfies (i)–(iii).

By induction we show that

$$(\forall i \in \omega)(p_i \leq b \rightarrow (p_i{}^i \text{ divides } c \wedge p_i{}^{i+1} \text{ does not divide } c))\,.$$

For $i = 0$ this is given by (i). Suppose that $i = j + 1$ and that $p_j{}^j$ divides c but $p_j{}^{j+1}$ does not. The desired conclusion follows from (ii) with $q = p_j$ and $r = p_i$, since $j < p_j{}^j \leq c$.

Now b is prime, and so $b = p_j$ for some j. Thus b^j divides c and b^{j+1} does not. By (iii), it follows that $j = a$. □

For natural numbers $a_0, \ldots, a_m$, let

$$\langle a_0, \ldots, a_m \rangle = p_0{}^{a_0+1} \cdot \ldots \cdot p_m{}^{a_m+1}\,.$$

For $m = -1$, let $\langle\ \rangle = 1$. Let Seq be the set of all a such that $a = \langle a_0, \ldots, a_m \rangle$ for some $m \geq -1$ and some $a_0, \ldots, a_m$. For elements a and b of ω, let

$$(a)_b = \mu n\, (p_b{}^{n+2} \text{ does not divide } a)\,.$$

Lemma 4.14. *(a) For each $m \in \omega$, the function*

$$(a_0, \ldots, a_{m-1}) \mapsto \langle a_0, \ldots, a_{m-1} \rangle$$

is representable in QE*. (b) The function $(a, b) \mapsto (a)_b$ is representable in* QE*. (c)* Seq *is representable in* QE*.*

Proof. (a) holds by closure under composition. For (b), apply the μ-operator to the characteristic function of the relation

$$p_b{}^{n+2} \text{ divides } a\,.$$

For (c), note that

$$a \in \mathrm{Seq} \;\leftrightarrow\; a > 0 \wedge (\forall i \leq a)(p_{i+1} \text{ divides } a \rightarrow p_i \text{ divides } a)\,. \qquad \square$$

For $a \in \omega$, let

$$\mathrm{lh}(a) = \mu n\,(p_n \text{ does not divide } a)\,.$$

For a and b elements of ω, let

$$a \lceil b = \mu n\,(a = 0 \vee (n \neq 0 \wedge (\forall j{<}b)(\forall k{<}a)(p_j{}^k \text{ divides } a \rightarrow p_j{}^k \text{ divides } n)))\,.$$

The following lemma follows easily from the definitions and earlier results.

Lemma 4.15. *The functions* lh *and $(a, b) \mapsto (a \lceil b)$ are representable in* QE*. For all $m \geq -1$ and all $a_0, \ldots, a_m$,*

(i) $\mathrm{lh}(\langle a_0, \ldots, a_m \rangle) = m + 1$*;*

(ii) $\langle a_0, \ldots, a_m \rangle \lceil b = \langle a_0, \ldots, a_{b-1} \rangle$ if $b \leq m + 1$.

For $n \in \omega$ and $h : {}^{n+1}\omega \to \omega$, let $\bar{h} : {}^{n+1}\omega \to \omega$ be given by

$$\bar{h}(a_1, \ldots, a_n, b) = \langle h(a_1, \ldots, a_n, 0), \ldots, h(a_1, \ldots, a_n, b-1) \rangle\,.$$

Lemma 4.16. *The set of functions representable in* QE *is closed under primitive recursion* (III)*.*

Proof. Let $h : {}^{n+1}\omega \to \omega$ be defined from $f : {}^{n}\omega \to \omega$ and $g : {}^{n+2}\omega \to \omega$ as in the statement of (III). Assume that f and g are representable in QE. We first show that $\bar{h}$ is representable:

$$\begin{aligned}\bar{h}(a_1, \ldots, a_n, b) \quad = \quad & \mu m\,(m \in \mathrm{Seq} \wedge \mathrm{lh}(m) = b \wedge \\ & \quad (\forall i < b)((i = 0 \wedge (m)_i = f(a_1, \ldots, a_n)) \vee \\ & \quad\; (\exists j < i)(i = j + 1 \wedge (m)_i = g(a_1, \ldots, a_n, j, (m)_j)))) \,.\end{aligned}$$

Now we note that

$$h(a_1, \ldots, a_n, b) = (\bar{h}(a_1, \ldots, a_n, b + 1))_b \,. \qquad \Box$$

Theorem 4.17. *Every recursive function is representable in* QE.

Proof. This follows from Lemmas 4.4, 4.5, 4.16, and 4.8. $\Box$

Our next goal is to show that various functions coding syntactical relations in languages such as $\mathcal{L}^{\mathrm{PAE}}$ are primitive recursive.

Lemma 4.18. *If* $t(v_1, \ldots, v_n)$ *is a term of* $\mathcal{L}^{\mathrm{PAE}}$*, then the function*

$$(a_1, \ldots, a_n) \mapsto (t(\mathbf{S}^{a_1}(0), \ldots, \mathbf{S}^{a_n}(0)))_{\mathcal{N}'}$$

is primitive recursive.

Proof. Successor and the constant function with value 0 are primitive recursive by (I). Addition, multiplication, and exponentiation are successively given by primitive recursion. For general terms, use composition and the I_i^n.

$\Box$

Lemma 4.19. *The functions* sg, pred*, and* $\dot{-}$ *are primitive recursive, where*

$$\begin{aligned}\mathrm{sg}(a) &= \begin{cases} 1 & \text{if } a > 0\,; \\ 0 & \text{if } a = 0\,; \end{cases} \\ \mathrm{pred}(a) &= \begin{cases} a - 1 & \text{if } a > 0\,; \\ 0 & \text{if } a = 0\,; \end{cases} \\ a \mathbin{\dot{-}} b &= \begin{cases} a - b & \text{if } a \geq b\,; \\ 0 & \text{if } a < b\,; \end{cases}\end{aligned}$$

Exercise 4.3. Prove Lemma 4.19.

Hint. Use primitive recursion.

Call a relation *primitive recursive* or *recursive* if its characteristic function is.

Lemma 4.20. *The set of all primitive recursive relations is closed under complement, intersection, and union. The relation $<$ is primitive recursive.*

Proof. Note that $K_{\neg R}(a_1, \dots, a_n) = 1 \dot{-} K_R(a_1, \dots a_n)$, that $K_{R \cap S}(a_1, \dots, a_n) = K_R(a_1, \dots, a_n) \cdot K_S(a_1, \dots, a_n)$, that $K_{R \cup S}(a_1, \dots, a_n) = \operatorname{sg}(K_R(a_1, \dots, a_n) + K_S(a_1, \dots, a_n))$, and that $K_<(a, b) = \operatorname{sg}(b \dot{-} a)$. □

Lemma 4.21. *The set of primitive recursive functions is closed under the two operations $f \mapsto g$ given by*

$$\begin{aligned} g(a_1, \dots, a_n, b) &= \sum_{b' < b} f(a_1, \dots, a_n, b')\,; \\ g(a_1, \dots, a_n, b) &= \prod_{b' < b} f(a_1, \dots, a_n, b')\,. \end{aligned}$$

(We consider the empty product to have value 1*.)*

Proof. We consider only the case of $\sum$. That of product is similar. We have

$$\begin{aligned} g(a_1, \dots, a_n, 0) &= 0\,; \\ g(a_1, \dots, a_n, \mathcal{S}(b)) &= g(a_1, \dots, a_n, b) + f(a_1, \dots, a_n, b)\,. \end{aligned}$$

Thus g comes by primitive recursion from functions that are primitive recursive if f is. □

Lemma 4.22. *The set of primitive recursive relations and functions is closed under bounded quantification.*

Proof. Let $R'(a_1, \dots, a_n) \leftrightarrow (\exists b < f(a_1, \dots, a_n))\, R(a_1, \dots, a_n, b)$. Then

$$K_{R'}(a_1, \dots, a_n) = \operatorname{sg}\left(\sum_{b < f(a_1, \dots, a_n)} K_R(a_1, \dots, a_n, b) \right).$$ □

Lemma 4.23. *The set of primitive recursive functions is closed under the* bounded μ-operator*, i.e., under $\langle f,g\rangle \mapsto h$, where*

$$h(a_1,\ldots,a_n) = \mu b\,(b = f(a_1,\ldots,a_n) \lor g(a_1,\ldots,a_n,b)=0)\,.$$

Exercise 4.4. Prove Lemma 4.23.

Lemma 4.24. *The relations and functions representable in* QE *by Lemmas 4.11, 4.12, 4.13, 4.14, and 4.15 are primitive recursive.*

Proof. The proofs of representability, with minor modifications, yield proofs of primitive recursiveness. The main thing to note is that the uses of the μ-operator in defining $(a)_b$, $a\lceil b$, and $\mathrm{lh}(a)$, are equivalent to the corresponding uses of the bounded μ-operator, with the bound function f in each case a constant function with value a. $\square$

Define $* : {}^2\omega \to \omega$ by

$$a * b = a \cdot \prod_{i<\mathrm{lh}(b)} p_{\mathrm{lh}(a)+i}{}^{(b)_i+1}\,.$$

The following lemma is evident.

Lemma 4.25. *The function $*$ is primitive recursive. For m and $n \geq -1$ and for any elments $a_0,\ldots,a_m$, $b_0,\ldots,b_n$ of ω,*

$$\prec a_0,\ldots,a_m\succ * \prec b_0,\ldots,b_n\succ = \prec a_0,\ldots,a_m,b_0,\ldots,b_n\succ\,.$$

For any $n \in \omega$ and any $f : {}^{n+1}\omega \to \omega$, define a function $(a_1,\ldots,a_n,b) \mapsto \ast_{i<b} f(a_1,\ldots,a_n,i)$ by

$$\begin{aligned} \ast_{i<0} f(a_1,\ldots,a_n,i) &= 1\,;\\ \ast_{i<b+1} f(a_1,\ldots,a_n,i) &= (\ast_{i<b} f(a_1,\ldots,a_n,i)) * f(a_1,\ldots,a_n,b)\,. \end{aligned}$$

The following lemma is also evident.

Lemma 4.26. *The function $(a_1,\ldots,a_n,b) \mapsto \ast_{i<b} f(a_1,\ldots,a_n,i)$ is primitive recursive if f is primitive recursive.*

Recall our official definition on page 21 of the logical symbols of our formal languages.

Fix a language $\mathcal{L}$ all of whose symbols are natural numbers: i.e., $\mathcal{L} = \langle f, p \rangle$ with all $f(m)$ and all $p(m)$ subsets of ω. Let us assume the following relations are primitive recursive:

$$\{(k,m) \mid k \in p(m)\}\,;$$
$$\{(k,m) \mid k \in f(m)\}\,.$$

Since we have not given the official definition of $\mathcal{L}^{\mathrm{PAE}}$, let us now declare:

$$\begin{array}{cc} \mathbf{0} & 13 \\ \mathbf{S} & 15 \\ + & 17 \\ \cdot & 19 \\ \mathbf{E} & 21 \\ < & 23 \end{array}$$

Note that our assumptions about $\mathcal{L}$ hold for $\mathcal{L}^{\mathrm{PAE}}$.

We assign numbers to finite sequences of symbols of $\mathcal{L}$ (to *expressions of* $\mathcal{L}$) by setting

$$\#(s_0, \ldots, s_n) = \langle s_0, \ldots, s_n \rangle\,.$$

When we talk of the $\#$ of a symbol s, we mean $\#(s)$, i.e., $\langle s \rangle$. We assign numbers to sequences of expressions (for example, to deductions) by

$$\#(\psi_0, \ldots, \psi_n) = \langle \#\psi_0, \ldots, \#\psi_n \rangle\,.$$

Lemma 4.27. *The following are primitive recursive:*

(1) the set of all #'s of variables;

(2) the set of all #'s of terms;

(3) the set of all #'s of atomic formulas;

(4) the set of all #'s of formulas.

Proof. (1) For $a \in \omega$, a is the $\#$ of a variable iff and only if

$$a \in \mathrm{Seq} \wedge \mathrm{lh}(a) = 1 \wedge 2 \text{ divides } (a)_0\,.$$

(2) Let f be the characteristic function of the set of all #'s of terms. We shall show that $\bar{f}$ is primitive recursive, from which it follows that f is

primitive recursive. Note first that $\bar{f}(0) = 1$. For any number a, a is the # of a term if and only if either a is the # of a variable or constant or

$$\begin{aligned}
&(\exists b)(\exists c)(b < a \,\wedge\, c < {p_a}^{a \cdot \mathrm{lh}(a)} \,\wedge\, c \in \mathrm{Seq} \,\wedge \\
&\qquad b \text{ is the } \# \text{ of a } \mathrm{lh}(c)\text{-place function symbol} \wedge \\
&\qquad (\forall i < \mathrm{lh}(c))((c)_i < a \,\wedge\, (c)_i \text{ is a term}) \,\wedge \\
&\qquad a = b * \#(() * (\textstyle\ast_{i<\mathrm{lh}(c)} (c)_i) * \#())) \,.
\end{aligned}$$

Because of the condition $(c)_i < a$, we can replace "$(c)_i$ is a term" by "$(\bar{f}(a))_{(c)_i} = 1$." Hence we can write $f(a)$ and so $\bar{f}(a+1)$ as a primitive recursive function of a and $\bar{f}(a)$. By (III), $\bar{f}$ is primitive recursive.

(3) is easy using (2).

The proof of (4) is similar in structure to that of (2). □

Lemma 4.28. *The set of all #'s of tautologies is primitive recursive.*

Proof. If ψ is a proper subformula of a formula φ, then $\#\psi < \#\varphi$. Using this fact, we can see that, for any $a \in \omega$, a is the # of a tautology if and only if a is the # of a formula and, for all $e < {p_a}^{2(a+1)}$, if

$$\begin{aligned}
&e \in \mathrm{Seq} \,\wedge\, \mathrm{lh}(e) = a + 1 \,\wedge \\
&\quad (\forall i \leq a)\, (e)_i \leq 1 \,\wedge \\
&\quad (\forall i \leq a)(\forall j < i)(i = \#(\neg) * j \,\rightarrow\, (e)_i = 1 \dot{-} (e)_j) \,\wedge \\
&\quad (\forall i \leq a)(\forall j < i)(\forall k < i)(i = \#(() * j * \#(\wedge) * k * \#()) \\
&\qquad\qquad\qquad\qquad\qquad \rightarrow\, (e)_i = (e)_j \cdot (e)_k) \,,
\end{aligned}$$

then $(e)_a = 1$. □

Lemma 4.29. *(1) There is a primitive recursive function* Sb *such that, if φ is a formula or a term, x is a variable, and t is a term, then*

$$\mathrm{Sb}(\#\varphi, \#(x), \#t) = \#\varphi(t)$$

where $\varphi(t)$ is the result of substituting t for the free occurrences of x in φ.

(2) There is a primitive recursive relation Fr *such that, if φ is a formula and x is a variable, then*

$$\mathrm{Fr}(\#\varphi, \#(x)) \leftrightarrow \; x \textit{ occurs free in } \varphi \,.$$

(3) The set of all #'s of sentences is primitive recursive.

(4) There is a primitive recursive relation Sbl *such that, if φ is a formula and x, t, and $\varphi(t)$ are as in (1), then*

$$\begin{aligned}
&\mathrm{Sbl}(\#\varphi, \#(x), \#t) \leftrightarrow \\
&\qquad \textit{no occurrence of a variable in } t \textit{ becomes bound in } \varphi(t).
\end{aligned}$$

The proof of the lemma will be a final examination problem.

Lemma 4.30. *(a) The set of all #'s of logical axioms is primitive recursive.*

(b) The set of all $(\#\varphi, \#\psi, \#\chi)$ such that χ follows from φ and ψ by Modus Ponens is primitive recursive.

(c) The set of all $(\#\varphi, \#\psi)$ such that ψ follows from φ by the Quantifier Rule is primitive recursive.

Proof. (a) We have already dealt with tautologies in Lemma 4.28. The identity axioms are easily handled using parts (2) and (3) of Lemma 4.27 and the function Sb. Quantifier Axioms are handled using Sbl and Sb.

(b) and (c) are proved in a straightforward manner, with Fr used for the latter. $\square$

Lemma 4.31. *Suppose that $\mathcal{L}$ extends $\mathcal{L}^{\mathrm{PA}}$. The set of #'s of axioms of* PA *is primitive recursive.*

Proof. There are finitely many axioms plus the induction schema. Instances of the latter are easily characterized using Sb. $\square$

A theory T in $\mathcal{L}$ is *recursively axiomatizable* if there is a set Σ of sentences such that

(i) $\{\#\sigma \mid \sigma \in \Sigma\}$ is recursive;

(ii) $T = \{\tau \mid \Sigma \models \tau\}$.

The notion of a *primitively recursively axiomatizable* theory is similarly defined, with "primitive recursive" replacing "recursive" in clause (i).

Remark. In fact, the class of recursively axiomatizable theories turns out to be the same as the class of primitively recursively axiomatizable theories.

Lemma 4.32. *Suppose that T is a primitively recursively axiomatizable theory in $\mathcal{L}$. Let Σ witness this fact. Then there is a primitive recursive relation* Pr *such that, for all a and $b \in \omega$,* $\mathrm{Pr}(a, b)$ *holds if and only if a is the # of a sentence τ and b is the # of a deduction of τ from Σ.*

Proof. The lemma follows easily from Lemma 4.30. $\square$

Theorem 4.33. *The functions reprensentable in* QE *are exactly the recursive functions.*

Proof. By Theorem 4.17, we need only show that every function representable in QE is recursive. Suppose $\varphi(v_1, \ldots, v_{n+1})$ represents $f : {}^n\omega \to \omega$ in QE. Let Pr be given by Lemma 4.32 for $T = \mathrm{QE}$. Note that the function

$$(a_1, \ldots, a_{n+1}) \mapsto \#\varphi(\mathbf{S}^{a_1}(\mathbf{0}), \ldots, \mathbf{S}^{a_{n+1}}(\mathbf{0}))$$

is primitive recursive, since the # of $\varphi(\mathbf{S}^{a_1}(\mathbf{0}), \ldots, \mathbf{S}^{a_{n+1}}(\mathbf{0}))$ is

$$\mathrm{Sb}(\ldots(\mathrm{Sb}(\#\varphi, \#(v_1), \#\mathbf{S}^{a_1}(\mathbf{0})), \ldots), \#(v_{n+1}), \#\mathbf{S}^{a_{n+1}}(\mathbf{0}))\,,$$

and since the function $a \mapsto \#\mathbf{S}^a(\mathbf{0})$ is easily seen to be primitive recursive. Define a recursive function $g : {}^n\omega \to \omega$ by

$$g(a_1, \ldots, a_n) = \mu b\, \mathrm{Pr}(\#\varphi(\mathbf{S}^{a_1}(\mathbf{0}), \ldots, \mathbf{S}^{a_n}(\mathbf{0}), \mathbf{S}^{(b)_0}(\mathbf{0})), (b)_1)\,.$$

For all $(a_1, \ldots, a_n)$,

$$f(a_1, \ldots, a_n) = (g(a_1, \ldots, a_n))_0\,. \qquad \square$$

We now know that the recursive functions have all the closure propeties of those representable in QE. (We could have directly proved these closure properties, as we did for the primitive recursive functions.) Thus we get the following lemma.

Lemma 4.34. *Lemma 4.32 continues to hold when the words "primitively" and "primitive" are deleted from its statement.*

Remark. By Lemma 4.34 and the proof of Lemma 4.33, any function representable in any recursively axiomatizable theory is recursive.

Lemma 4.35 (Fixed Point Lemma). *Let $\varphi(v_1)$ be a formula of $\mathcal{L}^{\mathrm{PAE}}$. There is a sentence σ such that*

$$\mathrm{QE} \models (\sigma \leftrightarrow \varphi(S^{\#\sigma}(\mathbf{0}))\,.$$

Proof. Let $\psi(v_1, v_2, v_3)$ represent in QE the primitive recursive function

$$(a, n) \mapsto \mathrm{Sb}(a, \#v_1, \#\mathbf{S}^n(\mathbf{0}))\,.$$

Note that, for any formula $\chi(v_1)$ and any $n \in \omega$, this function sends $(\#\chi, n)$ to $\#\chi(\mathbf{S}^n(\mathbf{0}))$.

Let $\chi(v_1)$ be the following formula:

$$(\forall v_3)(\psi(v_1, v_1, v_3) \to \varphi(v_3))\,.$$

Let $q = \#\chi(v_1)$.

Now let σ be the sentence

$$(\forall v_3)(\psi(\mathbf{S}^q(\mathbf{0}), \mathbf{S}^q(\mathbf{0}), v_3) \to \varphi(v_3))\,.$$

Note that σ is the result of replacing v_1 by $\mathbf{S}^q(\mathbf{0})$ in the formula $\chi(v_1)$. In other words, $\#\sigma$ is the value of the function represented by ψ on the argument (q, q). Hence

$$\mathrm{QE} \models (\forall v_3)(\psi(\mathbf{S}^q(\mathbf{0}), \mathbf{S}^q(\mathbf{0}), v_3) \;\leftrightarrow\; v_3 = \mathbf{S}^{\#\sigma}(\mathbf{0}))\,.$$

In particular,

$$\mathrm{QE} \models \psi(\mathbf{S}^q(\mathbf{0}), \mathbf{S}^q(\mathbf{0}), \mathbf{S}^{\#\sigma}(\mathbf{0}))\,.$$

Thus

$$\mathrm{QE} \models (\sigma \to \varphi(\mathbf{S}^{\#\sigma}(\mathbf{0}))\,.$$

But also

$$\mathrm{QE} \models (\forall v_3)(\psi(\mathbf{S}^q(\mathbf{0}), \mathbf{S}^q(\mathbf{0}), v_3) \;\to\; v_3 = \mathbf{S}^{\#\sigma}(\mathbf{0}))\,.$$

Therefore

$$\mathrm{QE} \models (\varphi(\mathbf{S}^{\#\sigma}(\mathbf{0})) \to \sigma)\,. \qquad \square$$

It is worth recording the following fact: Suppose $\psi(v_1, \ldots, v_n)$ represents in QE a relation R. Since $\mathfrak{N}' \models \mathrm{QE}$, we have that

$$(\forall a_1 \in \omega) \cdots (\forall a_n \in \omega)(R(a_1, \ldots, a_n) \leftrightarrow \mathfrak{N}' \models \psi[a_1, \ldots, a_n])\,.$$

Theorem 4.36. *Let T be a recursively axiomatizable theory in $\mathcal{L}^{\mathrm{PAE}}$ such that $\mathfrak{N}' \models T$. Then T is not complete.*

Proof. Let Pr be given by Lemma 4.34. Let ψ witness that Pr is representable in QE. Let $\varphi(v_1)$ be the formula

$$(\forall v_2)\neg\psi(v_1, v_2)\,.$$

Let σ be given be the Fixed Point Lemma.

One can think of σ as expressing its own unprovability in T. Indeed, by the observation preceding the theorem,

$$T \not\models \sigma \leftrightarrow \mathfrak{N}' \models \sigma\,.$$

If the consistent theory $T \models \sigma$ or $\models \neg\sigma$, then this contradicts the hypothesis that $\mathfrak{N}' \models T$. $\square$

Theorem 4.37. *Let T be any theory in $\mathcal{L}^{\mathrm{PAE}}$ such that $T \cup \mathrm{QE}$ is consistent. Then $\{\#\tau \mid \tau \in T\}$ is not recursive.*

Proof. Suppose for a contradiction that $\{\#\tau \mid \tau \in T\}$ is recursive. Let

$$T' = \{\tau \mid T \cup \mathrm{QE} \models \tau\}\,.$$

Let ρ be the conjunction of the finitely many axioms of QE. Then

$$\tau \in T' \;\leftrightarrow\; (\rho \rightarrow \tau) \in T\,,$$

so $\{\#\tau \mid \tau \in T'\}$ is recursive.

By Theorem 4.17, let $\psi(v_1)$ represent $\{\#\tau \mid \tau \in T'\}$ in QE. Let σ be given by the Fixed Point Lemma with $\neg\psi$ as φ.

Suppose first that $\sigma \notin T'$. Then

$$\mathrm{QE} \models \neg\psi(\mathbf{S}^{\#\sigma}(\mathbf{0}))\,.$$

But this implies that

$$\mathrm{QE} \models \sigma\,,$$

which in turn implies that $\sigma \in T'$.

Suppose then that $\sigma \in T'$. We successively get that $\mathrm{QE} \models \psi(\mathbf{S}^{\#\sigma}(\mathbf{0}))$, that $\mathrm{QE} \models \neg\sigma$, and that $\neg\sigma \in T'$. $\square$

Corollary 4.38 (Church's Theorem). *The set of all #'s of valid sentences in $\mathcal{L}^{\mathrm{PAE}}$ is not recursive.*

Corollary 4.39. *If T be a recursively axiomatizable theory in $\mathcal{L}^{\mathrm{PAE}}$ such that $T \cup \mathrm{QE}$ is consistent, then T is not complete.*

Proof. It suffices to prove that if Σ is a set of sentences such that $\{\#\sigma \mid \sigma \in \Sigma\}$ is recursive and the theory $T = \{\tau \mid \Sigma \models \tau\}$ is complete, then $\{\#\tau \mid \tau \in T\}$ is recursive. For this, fix Σ and let Pr be given by Lemma 4.34. Assume that T is is complete. Define $g : \omega \rightarrow \omega$ by setting $g(a) = 0$ if a is not the # of a sentence and otherwise setting

$$g(a) = \mu b\,(\mathrm{Pr}(a,b) \,\vee\, \mathrm{Pr}(\#(\neg\ ) * a, b))\,.$$

Since T is complete, g is a recursive function. Moreover, for any $a \in \omega$,

$$a \in \{\#\tau \mid \tau \in T\} \;\leftrightarrow\; (g(a) \neq 0 \,\wedge\, \mathrm{Pr}(a, g(a)))\,. \qquad \square$$

A theory T in $\mathcal{L}$ is *recursively decidable* if $\{\#\tau \mid \tau \in T\}$ is recursive. Otherwise T is *recursively undecidable*. Thus Church's Theorem shows that the set of valid sentences of $\mathcal{L}^{\mathrm{PAE}}$ is not recursively decidable. (Church's Theorem is actully more general, holding for, say, any language with a two-place relation symbol.) According to *Church's Thesis*, the recursive functions are exactly the effectively computable functions. Granted Church's Thesis, decidability and recursive decidability are the same.

To eliminate exponentiation and get incompleteness theorems for PA, we shall use the following number-theoretic result.

Lemma 4.40 (Chinese Remainder Theorem). *Let the positive integers $d_0, \ldots, d_n$ be relatively prime. Let $a_i < d_i$ for each $i \leq n$. Then there is a c such that, for each $i \leq n$, a_i is the remainder when c is divided by d_i.*

Proof. For any $c \in \omega$, let $\mathbf{F}(c) = (r_0, \ldots, r_n)$, where each r_i is the remainder when c is divided by d_i.

Suppose c_1 and c_2 are distinct numbers smaller than $\prod_{i \leq n} d_i$. If $\mathbf{F}(c_1) = \mathbf{F}(c_2)$, then each d_i divides $|c_1 - c_2|$ and so, since the d_i are relatively prime, $\prod_{i \leq n} d_i$ divides $|c_1 - c_2|$. This contradiction shows that $\mathbf{F}(c_1) \neq \mathbf{F}(c_2)$.

Thus $\mathbf{F}(c)$ takes on $\prod_{i \leq n} d_i$ distinct values for $c < \prod_{i \leq n} d_i$. But each of these values is of the form $(r_0, \ldots, r_n)$ with each $r_i < d_i$. There are only $\prod_{i \leq n} d_i$ such $(r_0, \ldots, r_n)$, so one of the $\mathbf{F}(c)$ must be $(a_0, \ldots, a_n)$. $\square$

Lemma 4.41. *For any positive integer m, the numbers $1 + (i + 1) \cdot m!$, $i \leq m$, are relatively prime.*

Proof. Let i and j be distinct numbers $\leq m$. Suppose that some prime p divides both $1 + (i + 1) \cdot m!$ and $1 + (j + 1) \cdot m!$, with i and $j \leq m$. Then p divides $|i - j| \cdot m!$. Since p cannot divide $m!$, it follows that p must divide $|i - j|$. But $|i - j| \leq m$, and thus we have the contradiction that p divides $m!$.

$\square$

For elements c, d, and i of ω, let $r(c, d, i)$ be the remainder when c is divided by $1 + (i + 1) \cdot d$.

Order the set of all pairs (a, b) of natural numbers first by $\max\{a, b\}$ and then lexicographically. For pairs (a, b), let $n(a, b)$ be the number of pairs preceding (a, b) in this ordering. Define $q_1 : \omega \to \omega$ and $q_2 : \omega \to \omega$ by setting $q_1(n(a, b)) = a$ and $q_2(n(a, b)) = b$.

Let Q be the set of consequences in $\mathcal{L}^{\mathrm{PA}}$ of Axioms 1–8.

Lemma 4.42. *The functions r, n, q_1, and q_2 are representable in* Q.

Proof. Note that all our lemmas before Lemma 4.13 continue to hold if we replace $\mathcal{L}^{\mathrm{PAE}}$ by $\mathcal{L}^{\mathrm{PA}}$ and QE by Q. We have

$$\begin{aligned} r(c,d,i) &= \mu b\,(\exists e \le c)\, c = (1+(i+1)\cdot d)\cdot e + b\,; \\ \max\{a,b\} &= \mu c\,(a \le c \wedge b \le c)\,; \\ n(a,b) &= (\max\{a,b\})^2 + a + b\cdot K_{\le}(b,a)\,; \\ q_1(c) &= \mu a\,(\exists b \le c)\, n(a,b) = c\,; \\ q_2(c) &= \mu b\,(\exists a \le c)\, n(a,b) = c\,. \end{aligned}$$

□

Lemma 4.43. *For any natural numbers n and $a_0, \ldots, a_n$, there are c and d such that*

$$(\forall i \le n)\, r(c,d,i) = a_i\,.$$

Proof. Given n and $a_0, \ldots a_n$, let $m = \max\{n, a_0, \ldots, a_n\}$. Let $d = m!$. Since the $1+(i+1)\cdot d$ are relatively prime, let c be given by the Chinese Remainder Theorem. (Note that each $a_i < 1+(i+1)\cdot d$.) □

Lemma 4.44. *Exponentiation is representable in* Q.

Proof. Define functions $f : {}^2\omega \to \omega$ and $E^* : {}^2\omega \to \omega$ by

$$\begin{aligned} f(m,i) &= r(q_1(m), q_2(m), i)\,; \\ E^*(a,b) &= \mu m\,(f(m,0) = 1 \wedge (\forall i \le b)\, f(m,i+1) = f(m,i)\cdot a)\,. \end{aligned}$$

Both f and E^* are representable in Q. Moreover, we have that

$$(\forall a \in \omega)(\forall b \in \omega)(\forall i \le b)\, f(E^*(a,b),i) = a^i\,.$$

Thus $a^b = f(E^*(a,b),b)$ for all a and b. □

Theorem 4.45. *All previous lemmas, theorems, and corollaries of Section 4 hold with $\mathcal{L}^{\mathrm{PA}}$ replacing $\mathcal{L}^{\mathrm{PAE}}$ and* Q *replacing* QE.

Theorem 4.46. PA *is incomplete and recursively undecidable. Moreover all recursively axiomatizable extensions of* PA *are incomplete, and all consistent extensions of* PA *are recursively undecidable.*

Proof. This follows from Theorems 4.45, Theorem 4.36 or Corollary 4.39, and Theorem 4.37. $\square$

Theorem 4.36, Theorem 4.37, Corollary 4.39, and Theorem 4.46 are all versions of Gödel's First Incompleteness Theorem. We end this section with a brief sketch of Gödel's Second Incompleteness Theorem.

Let Pr be given by Lemma 4.34 for some recursively axiomatizable T in $\mathcal{L}^{\mathrm{PA}}$ such that $\mathrm{Q} \subseteq T$. Let ψ witness that Pr is representable in Q. Let σ be given by the Fixed Point Lemma, with QE_0 replacing QE and with $(\forall v_2)\neg\psi(v_1, v_2)$ as $\varphi(v_1)$. Thus $T \not\models \sigma$ if and only if σ is true in $\mathfrak{N}$.

Suppose that σ is false in $\mathfrak{N}$, i.e., suppose that $T \models \sigma$. Then there is a $b \in \omega$ such that $\mathrm{Pr}(\#\sigma, b)$. For any such b,

$$\mathrm{Q} \models \psi(\mathbf{S}^{\#\sigma}(\mathbf{0}), \mathbf{S}^{b}(\mathbf{0}))\,.$$

Hence

$$\mathrm{Q} \models (\exists v_2)\psi(\mathbf{S}^{\#\sigma}(\mathbf{0}), v_2)\,.$$

In other words,

$$\mathrm{Q} \models \neg\varphi(\mathbf{S}^{\#\sigma}(\mathbf{0}))\,.$$

But then $\mathrm{Q} \models \neg\sigma$, and so $T \models \neg\sigma$. Therefore T is inconsistent.

The argument of the last paragraph shows that if T is consistent then σ is true in $\mathfrak{N}$. The converse of this fact also holds: If σ is true, then $T \not\models \sigma$, and so T is consistent. Thus σ is true in $\mathfrak{N}$ if and only if T is consistent.

Using the formula ψ and formulas representing the set of all $\#$'s of sentences and the function $a \mapsto \#(\neg) * a$, we can construct a sentence $\ulcorner\mathrm{Con}\ T\urcorner$ of $\mathcal{L}^{\mathrm{PA}}$ that we may think of as expressing the consistency of T. Our argument then establishes the truth of

$$\sigma \leftrightarrow \ulcorner\mathrm{Con}\ T\urcorner.$$

Now comes the sketchy part of our discussion. If we have chosen natural representing formulas, then we can show that

$$\mathrm{PA} \models \sigma \leftrightarrow \ulcorner\mathrm{Con}\ T\urcorner.$$

This is essentially because our basic tool in our (presumably set theoretic) proof of (the set theoretic version of) this sentence was induction.

Now suppose that T is PA. Since PA is consistent, $\mathrm{PA} \not\models \sigma$. But then

$$\mathrm{PA} \not\models \ulcorner\mathrm{Con}\ \mathrm{PA}\urcorner.$$

In other words, the consistency of PA implies that the number theoretic version of the consistency of PA is not provable in PA.

The argument establishes that any consisent, recursively axiomatizable extension of PA cannot prove the number-theoretic sentence expressing its own consistency. This result can easily be extended to theories in which PA is interpretable. For example, one cannot prove in ZFC, if ZFC is consistent, the set-theoretic formulation of the consistency of ZFC.

5 Recursion Theory

Fix $n \in \omega \setminus \{0\}$. To get a useful enumeration of the recursive functions, we do a uniform version of the construction of the proof of Theorem 4.33. Let $\mathrm{Pr}(a, d)$ hold if and only if d is the $\#$ of a deduction from the axioms of Q of a sentence σ of $\mathcal{L}^{\mathrm{PA}}$ such that $a = \#\sigma$. Define $T_n \subseteq {}^{n+2}\omega$ by letting $T_n(e, a_1, \ldots, a_n, d)$ hold if and only if

(i) For some formula $\varphi(v_1, \ldots, v_{n+1})$ of $\mathcal{L}^{\mathrm{PA}}$, $\#\varphi = e$;

(ii) $\mathrm{Pr}(\#\varphi(\mathbf{S}^{a_1}(\mathbf{0}), \ldots, \mathbf{S}^{a_n}(\mathbf{0}), \mathbf{S}^{(d)_0}(\mathbf{0})), (d)_1)$;

(iii) d is the smallest number satisfying (i) and (ii).

Define $U : \omega \to \omega$ by setting $U(d) = (d)_0$.

Theorem 5.1. *(a) For each $n \geq 1$, T_n is primitive recursive.*

(b) The function U is primitive recursive.

(c) If $n \geq 1$ and $f : {}^n\omega \to \omega$ is recursive, then there is an $e \in \omega$ such that, for all numbers $a_1, \ldots, a_n$,

$$f(a_1, \ldots, a_n) = U(\mu d\, T_n(e, a_1, \ldots, a_n, d)) .$$

(d) Every total (i.e., totally defined) function in this form is recursive.

Proof. For (a), note that clause (ii) is equivalent with

$$\mathrm{Pr}(\mathrm{Sb}(\ldots(\mathrm{Sb}(e, \#v_1, \#\mathbf{S}^{a_1}(\mathbf{0})), \ldots), \#v_{n+1}, \#\mathbf{S}^{(d)_0}(\mathbf{0})), (d)_1) .$$

For (c), let φ represent f in Q and let $e = \#\varphi$. (d) follows from (a) and (b). $\square$

A *partial (number-theoretic) function of n variables* is an $f : A \to \omega$ where $A \subseteq {}^n\omega$.

A partial function of n variables is *partial recursive* if there are recursive g and h such that

$$f(a_1, \ldots, a_n) \simeq h(\mu b\, g(a_1, \ldots, a_n, b) = 0) ,$$

where "$x \simeq y$" means "x and y are defined and equal or both are undefined."

Lemma 5.2. *For each n and e, the partial function f given by*

$$f(a_1, \ldots, a_n) \simeq U(\mu d\, T_n(e, a_1, \ldots, a_n, d))$$

is partial recursive.

Lemma 5.3. *If f is a partial recursive function of n variables, then there is an e such that, for all $a_1, \ldots, a_n$,*

$$f(a_1, \ldots, a_n) \simeq U(\mu d\, T_n(e, a_1, \ldots, a_n, d)) .$$

Proof. Let g and h witness that f is partial recursive. Let $\varphi(v_1, \ldots, v_{n+2})$ and $\psi(v_1, v_2)$ represent g and h respectively in Q. Let $\chi(v_1, \ldots, v_{n+1})$ be

$$(\exists z)(\varphi(v_1, \ldots, v_n, z, \mathbf{0}) \wedge (\forall z')(z' < z \rightarrow \neg\varphi(v_1, \ldots, v_n, z', \mathbf{0})) \wedge \psi(z, v_{n+1})) ,$$

for appropriate variables z and z'. It is easy to see that the sentence $\chi(\mathbf{S}^{a_1}(\mathbf{0}), \ldots, \mathbf{S}^{a_n}(\mathbf{0}), \mathbf{S}^{c}(\mathbf{0}))$ is provable in Q if and only if $c \simeq f(a_1, \ldots, a_n)$. (The main point is that only sentences true in $\mathfrak{N}$ are provable in Q.) Thus we can let $e = \#\chi$. $\square$

Theorem 5.4. *The partial recursive functions of n variables are exactly the functions $\{e\}_n$, where*

$$\{e\}_n(a_1, \ldots, a_n) \simeq U(\mu d\, T_n(e, a_1, \ldots, a_n, d)) .$$

Exercise 5.1. Define an operation of composition for partial functions and prove that the partial recursive functions are closed under composition.

A subset A of ω is *recursively enumerable* (*r.e.*) if A is the domain of a partial recursive function.

Theorem 5.5. *If $A \subseteq \omega$, then A is r.e. if and only if A is either empty or the range of a recursive function, where the function can be taken to be of one argument.*

Proof. Suppose the A is r.e. Then there is an e such that $A = \{a \mid (\exists d)\, T_1(e, a, d)\}$. Suppose that $A \neq \emptyset$. Let $a \in A$. Define a recursive g by setting

$$g(b) = \begin{cases} (b)_0 & \text{if } T_1(e, (b)_0, (b)_1) ; \\ a & \text{otherwise.} \end{cases}$$

Now suppose that $A = \text{range}\,(\tilde{g})$ with $\tilde{g}$ recursive. For $b \in \omega$, let

$$f(b) \simeq \mu c\, \tilde{g}((c)_1, \ldots, (c)_n)) = b .$$

Clearly $A = \text{domain}\,(f)$. To see that f is partial recursive, define g and h by:

$$\begin{aligned} g(b, c) &= (\tilde{g}((c)_1, \ldots, (c)_n)) \dot{-} b) + (b \dot{-} \tilde{g}((c)_1, \ldots, (c)_n) ; \\ h(a) &= a . \end{aligned}$$

It is easy to see that there is a partial recursive function with domain $\emptyset$: Note that, e.g., $\{0\}_1 = \emptyset$. $\square$

Theorem 5.6. *A subset A of ω is recursive if and only if both A and $\neg A$ are r.e.*

Proof. Suppose first that A is recursive. Define g and g' by setting

$$\begin{aligned} g(a) &\simeq \mu b\, K_A(a) = 1\,; \\ g'(a) &\simeq \mu b\, K_A(a) = 0\,. \end{aligned}$$

g and g' witness that A and $\neg A$ respectively are r.e.

For the converse, suppose that $A = \{a \mid (\exists d)\, T_1(e, a, d)\}$ and that $\neg A = \{a \mid (\exists d)\, T_1(e', a, d)\}$. Then

$$K_A(a) = K_{T_1}(e, a, \mu d\,(T_1, (e, a, d) \,\vee\, T_1(e', a, d)))\,. \qquad \square$$

Let $\mathcal{K} = \{e \mid (\exists d)\, T_1(e, e, d)\}$.

Theorem 5.7. *The set $\mathcal{K}$ is r.e. but not recursive.*

Proof. $\mathcal{K}$ is the domain of the partial recursive function f given by $f(e) \simeq U(\mu d\, T_1(e, e, d))$.

Suppose that $\mathcal{K}$ is recursive. Then $\neg\mathcal{K}$ is r.e., and so there is an e such that $\neg\mathcal{K} = \text{domain}\,(\{e\}_1)$. But then

$$e \in \mathcal{K} \leftrightarrow (\exists d)\, T_1(e, e, d) \leftrightarrow e \notin \mathcal{K}\,. \qquad \square$$

Remark. An obvious and important fact that we have failed to mention explicitly is that, for all $n \in \omega$, the partial function f of $n+1$ variables given by $f(e, a_1, \ldots, a_n) \simeq U(\mu d\, T_n(e, a_1, \ldots, a_n, d))$ is partial recursive.

Theorem 5.8 (s-m-n Theorem). *For any positive integers m and n, there is a recursive function S^m_n such that, for all $e, a_1, \ldots, a_m, b_1, \ldots, b_n$,*

$$\{e\}_{m+n}(a_1, \ldots, a_m, b_1, \ldots, b_n) \simeq \{S^m_n(e, a_1, \ldots, a_m)\}_n(b_1, \ldots, b_n)\,.$$

Proof. The idea of the proof is simple. In the case that matters, when e is the number of a formula $\varphi(v_1, \ldots, v_{m+n+1})$, then we would like to set $S^m_n(e, a_1, \ldots, a_m) = \#\varphi(\mathbf{S}^{a_1}(\mathbf{0}), \ldots, \mathbf{S}^{a_m}(\mathbf{0}), v_1, \ldots, v_{n+1})$. But, for $1 \leq i \leq n+1$, some occurrences of v_i that replace free occurrences of v_{m+i} may be bound. For this reason, we need to change the bound occurrences of these v_i to occurrences of other variables before we insert the v_i, and even this step requires preparation.

First let Let $f_n^m(e, a_1, \dots, a_m) =$

$$\underbrace{\mathrm{Sb}(\cdots\mathrm{Sb}(}_{n+1}\underbrace{\mathrm{Sb}(\cdots\mathrm{Sb}(}_{m}\,e\,\underbrace{,\#v_1,\#\mathbf{S}^{a_1}(\mathbf{0}))\cdots,\#v_m,\#\mathbf{S}^{a_m}(\mathbf{0})),}_{m}$$

$$\underbrace{\#v_{m+1},\#v_{e+1})\cdots,\#v_{m+n+1},\#v_{e+n+1})}_{n+1}.$$

Next let

$$g_n(e,c,i) = \begin{cases} (c)_i + 2(e+n+1) & \text{if } (c)_i \text{ is even and } 2 \le (c)_i \le 2(n+1); \\ (c)_i & \text{otherwise.} \end{cases}$$

Then let

$$h_n(e,c) = \prod_{i<\mathrm{lh}(a)} p_i^{g_n(e,c,i)+1}$$

and let

$$k_n^m(e,a_1,\dots,a_m) = h_n(e, f_n^m(e,a_1,\dots,a_m)).$$

Finally let

$$S_n^m(e,a_1,\dots,a_m) = \mathrm{Sb}(k_n^m(e,a_1,\dots,a_m),\#v_{e+1},\#v_1)\cdots,\#v_{e+n+1},\#v_{n+1})$$

if e is the # of a formula $\varphi(v_1,\dots,v_{m+n+1})$, and let $S_n^m(e,a_1,\dots,a_m) = 0$ otherwise.

To see how the definition works, note that if $e = \#\varphi(v_1,\dots,v_{m+n+1})$, then

$$f_n^m(e,a_1,\dots,a_m) = \#\varphi(\mathbf{S}^{a_1}(\mathbf{0}),\dots,\mathbf{S}^{a_m}(\mathbf{0}),v_{e+1},\dots,v_{e+n+1}).$$

In this case, $k_n^m(e,a_1,\dots,a_m)$ is the number of a formula we shall call $\psi(\mathbf{S}^{a_1}(\mathbf{0}),\dots,\mathbf{S}^{a_m}(\mathbf{0}),v_{e+1},\dots,v_{e+n+1})$, the formula that is gotten from $\varphi(\mathbf{S}^{a_1}(\mathbf{0}),\dots,\mathbf{S}^{a_m}(\mathbf{0}),v_{e+1},\dots,v_{e+n+1})$ by replacing all occurrences of v_i by occurrences of $v_{e+n+1+i}$ for $1 \le i \le n+1$. The replaced occurrences of v_i are bound occurrences, since these are the only occurrences of v_i. Finally,

$$S_n^m(e,a_1,\dots,a_m) = \#\psi(\mathbf{S}^{a_1}(\mathbf{0}),\dots,\mathbf{S}^{a_m}(\mathbf{0}),v_1,\dots,v_{n+1}).$$

□

For subsets A and B of ω, we say that A is *many-one reducible* to B ($A \le_m B$) if there is a recursive f such that

$$(\forall a \in \omega)(a \in A \leftrightarrow f(a) \in B).$$

From now on, we shall usually write $\{e\}$ for $\{e\}_1$.

Theorem 5.9. *Let $\mathcal{H} = \{b \in \omega \mid \{(b)_0\}((b)_1) \text{ is defined}\}$. Then $\mathcal{H}$ is r.e., $\mathcal{K} \leq_m \mathcal{H}$, and $\mathcal{H} \leq_m \mathcal{K}$.*

Proof. $\mathcal{H}$ is obviously r.e.

Let $f(e) = \prec e, e \succ$. Then, for any $e \in \omega$,

$$e \in \mathcal{K} \leftrightarrow \{e\}(e) \text{ is defined} \leftrightarrow \prec e, e \succ \in \mathcal{H}\,,$$

so $\mathcal{K} \leq_m \mathcal{H}$.

To show that $\mathcal{H} \leq_m \mathcal{K}$, we use the *s-m-n* Theorem. Define g by

$$g(b, a) \simeq \{(b)_0\}((b)_1)\,.$$

The partial function g is partial recursive, since

$$g(b, a) \simeq U(\mu d\,(T_1((b)_0, (b)_1, d))\,.$$

Hence there is an $e \in \omega$ such that

$$(\forall b)(\forall a)\, g(b, a) \simeq \{e\}_2(b, a)\,.$$

Set $f(b) = S^1_1(e, b)$ for $b \in \omega$. We have that

$$\{f(b)\}(a) \simeq \{S^1_1(e, b)\}(a) \simeq \{e\}_2(b, a) \simeq g(b, a)\,.$$

Suppose that $b \in \mathcal{H}$. Then $\{(b)_0\}((b)_1)$ is defined. Hence $g(b, a)$ is defined for every a, and so $\{f(b)\}(a)$ is defined for every a. In particular, $\{f(b)\}(f(b))$ is defined, and this means that $f(b) \in \mathcal{K}$.

Now suppose that $b \notin \mathcal{H}$. Then $\{(b)_0\}((b)_1)$ is undefined. Thus $\{f(b)\}$ is the completely undefined function, so $f(b) \notin \mathcal{K}$. □

Theorem 5.10. *Let $A \subseteq \omega$ be r.e. Then $A \leq_m \mathcal{H}$ and so $A \leq_m \mathcal{K}$.*

Proof. Let $A = \text{domain}(\{e\})$. Define f by setting $f(n) = \prec e, n \succ$. Then, for all n,

$$n \in A \;\leftrightarrow\; \{e\}(n) \text{ is defined } \;\leftrightarrow\; \prec e, n \succ \in \mathcal{H}\,.$$ □

The *s-m-n* Theorem implies that if g is a partial recursive function of $m + n$ variables, then there is a recursive f such that

$$\{f((a_1, \ldots, a_m)\}_n(b_1, \ldots, b_n) \simeq g(a_1, \ldots, a_m, b_1, \ldots, b_n)\,,$$

for all $a_1, \ldots, a_m, b_1, \ldots, b_n$. From now on we shall use this consequence of the *s-m-n* Theorem directly.

Theorem 5.11 (Recursion Theorem). *For all $m \in \omega$ and all recursive $f : \omega \to \omega$, there is an $n \in \omega$ such that $\{n\}_m = \{f(n)\}_m$.*

Proof. Define g by

$$g(u, a_1, \ldots, a_m) \simeq \{\{u\}(u)\}_m(a_1, \ldots, a_m)\,.$$

It is easy to see that g is partial recursive, so the s-m-n Theorem gives a recursive h such that, for all $u, a_1, \ldots, a_m$,

$$g(u, a_1, \ldots, a_m) \simeq \{h(u)\}_m(a_1, \ldots, a_m)\,.$$

Let $\{v\} = f \circ h$, the composition of f and h. Let $n = h(v)$. We have that

$$\begin{aligned}
\{n\}_m(a_1, \ldots, a_m) &\simeq \{h(v)\}_m(a_1, \ldots, a_m) \\
&\simeq g(v, a_1, \ldots, a_m) \\
&\simeq \{\{v\}(v)\}_m(a_1, \ldots, a_m) \\
&\simeq \{f(h(v))\}_m(a_1, \ldots, a_m) \\
&\simeq \{f(n)\}_m(a_1, \ldots, a_m)\,.
\end{aligned}$$

□

Theorem 5.12 (Uniform Recursion Theorem). *For each $m \in \omega$, there is a recursive function r_m such that, for all $e \in \omega$,*

$$\{e\} \text{ is total } \to \{r_m(e)\}_m = \{\{e\}(r_m(e))\}_m\,.$$

Proof. Define h as in the proof of Theorem 5.11. By the s-m-n Theorem, let v be a recursive function such that

$$(\forall e)(\forall n)\{v(e)\}(n) \simeq (\{e\} \circ h)(n)\,.$$

For each e, set $r_m(e) = h(v(e))$. □

For $e \in \omega$, let $W_e = \text{domain}\,(\{e\})$. Note that $\mathcal{K} = \{e \mid e \in W_e\}$.

An r.e. set C is *creative* if there is a recursive function f such that

$$(\forall e \in \omega)(W_e \cap C = \emptyset \to f(e) \notin W_e \cup C)\,.$$

If C is creative, then C is not recursive, for $f(e)$ witnesses that $\neg C \neq W_e$ whenever $W_e \subseteq \neg C$. (We write $\neg C$ for $\omega \setminus C$.)

The set $\mathcal{K}$ is witnessed creative by the identity function, for

$$W_e \cap \mathcal{K} = \emptyset \Rightarrow e \notin W_e \Rightarrow e \notin \mathcal{K}\,.$$

Theorem 5.13. *If C is creative and A is r.e., then $A \leq_m C$.*

Proof. Let f witness that C is creative, and let A be r.e. Define h by

$$h(a,b,c) \simeq \begin{cases} 0 & \text{if } a \in A \text{ and } c = f(b)\,; \\ \text{undefined} & \text{otherwise.} \end{cases}$$

It is easy to show that h is partial recursive. By applications of the s-m-n Theorem, let p and q be recursive and such that

$$\begin{aligned} h(a,b,c) &\simeq \{p(a,b)\}(c)\,; \\ p(a,b) &= \{q(a)\}(b)\,. \end{aligned}$$

Note that, for all a and b,

$$W_{p(a,b)} = \begin{cases} \{f(b)\} & \text{(the singleton) if } a \in A\,; \\ \emptyset & \text{otherwise}\,. \end{cases}$$

Let $r = r_1$. By the Uniform Recursion Theorem, we have for all a that

$$\begin{aligned} \{r(q(a))\} &= \{\{q(a)\}(r(q(a)))\} \\ &= \{p(a, r(q(a)))\}\,. \end{aligned}$$

Hence, for all a, $W_{r(q(a))} = W_{p(a,r(q(a)))}$.

We show that $f \circ r \circ q$ witnesses that $A \leq_m C$. Note first that

$$\begin{aligned} a \in A &\rightarrow W_{p(a,r(q(a)))} = \{f(r(q(a)))\} \\ &\rightarrow W_{r(q(a))} = \{f(r(q(a)))\} \\ &\rightarrow f(r(q(a))) \in C\,. \end{aligned}$$

(Since f witnesses that C is creative, the next-to-last line implies that $W_{r(q(a))} \cap C \neq \emptyset$. This and the next-to-last line imply the last line.) Note finally that

$$\begin{aligned} a \notin A &\rightarrow W_{p(a,r(q(a)))} = \emptyset \\ &\rightarrow W_{r(q(a))} = \emptyset \\ &\rightarrow f(r(q(a))) \notin C\,. \end{aligned}$$

(The last implication holds because f witnesses that C is creative.) $\square$

Exercise 5.2. The *join* of subsets A and B of ω is

$$\{2n \mid n \in A\} \cup \{2n+1 \mid n \in B\}\,.$$

Prove that the join of A and B is a $\leq_m$-least upper bound for A and B.

Exercise 5.3. (a) Show that if A is r.e. and $A \leq_m \neg A$ then A is recursive.

(b) Prove that the hypothesis that A is r.e. cannot be dropped from (a). *Hint.* Consider the join of a set and its complement.

Exercise 5.4. A subset A of ω is a *many-one complete r.e. set* if A is r.e. and, for all r.e. B, $B \leq_m A$. Thus all creative sets are many-one complete r.e. sets. Prove that $\{e \in \omega \mid W_e \neq \emptyset\}$ is a many-one complete r.e. set.

Exercise 5.5. Let C be creative. Show that there is a recursive f such that

$$(\forall e \in \omega)(f(e) \in W_e \cap C \lor f(e) \notin W_e \cup C)\,.$$

Hint. Let $\bar{f}$ witness that C is creative. Use the s-m-n Theorem to define a recursive p such that, for all a and b,

$$W_{p(a,b)} = W_a \cap \{\bar{f}(b)\}\,.$$

Now use the s-m-n Theorem and the Uniform Recursion Theorem to get a recursive s such that, for all a,

$$W_{s(a)} = W_{p(a,s(a))}\,.$$

Let $f = \bar{f} \circ s$.

Theorem 5.14. *If C is a many-one complete r.e. set, then C is creative.*

Proof. Let g witness that $\mathcal{K} \leq_m C$. By the s-m-n theorem, let h be recursive and such that

$$(\forall e)(\forall a)\,\{h(e)\}(a) \simeq \{e\}(g(a))\,.$$

Note that, for all e, $W_{h(e)}$ is the preimage under g of W_e.

Let $f = g \circ h$. To show that f witnesses that C is creative, let e be such that $W_e \cap C = \emptyset$. Taking preimages under g, we get that $W_{h(e)} \cap \mathcal{K} = \emptyset$. By the definition of $\mathcal{K}$, this implies that $h(e) \notin W_{h(e)} \cup \mathcal{K}$. But then $g(h(e)) \notin W_e \cup C$.

$\square$

Theorem 5.15. *For all m and n, there is a one-one function S^m_n that witnesses the truth of the s-m-n Theorem.*

Proof. Fix m and n. Let $\bar{S}^m_n$ have the property required of S^m_n in the statement of the s-m-n Theorem. Define $h : {}^{m+1}\omega \to \omega$ by setting

$$h(a_0, \ldots, a_m) = \#(\mathbf{S}^{a_0}(\mathbf{0}) = \mathbf{S}^{a_0}(\mathbf{0}) \wedge (\cdots \wedge \mathbf{S}^{a_m}(\mathbf{0}) = \mathbf{S}^{a_m}(\mathbf{0}))\cdots) .$$

It is easy to see that h is a one-one recursive function and that all the values of h are #'s of valid sentences of $\mathcal{L}^{\mathrm{PA}}$. Define S^m_n by setting

$$S^m_n(e, a_1, \ldots, a_m) = \#\langle(\rangle * h(e, a_1, \ldots, a_m) * \#\langle\wedge\rangle * \bar{S}^m_n(e, a_1, \ldots, a_m) * \#\langle)\rangle .$$

□

Theorem 5.16. *For each $m \in \omega$, there is a one-one function r_m that witnesses the truth of the Uniform Recursion Theorem.*

Proof. Given m, define functions h and v, as in the proof of Theorem 5.12, using one-one functions S^1_m and S^1_1. The h and v so defined are one-one. Hence $r_m = h \circ v$ is also one-one. □

Theorem 5.17. *If C is creative, then there is a one-one function witnessing that C is creative.*

Proof. Define a partial recursive function g by

$$g(e, n, y) \simeq \begin{cases} y & \text{if } (\exists i < \mathrm{lh}(n))\, y = (n)_i\, ; \\ \{e\}(y) & \text{otherwise} . \end{cases}$$

Let p be recursive and such that

$$(\forall e)(\forall n)(\forall y)\, \{p(e, n)\}(y) \simeq g(e, n, y) .$$

Thus

$$(\forall e)(\forall n)\, W_{p(e,n)} = W_e \cup \{(n)_0, \ldots, (n)_{\mathrm{lh}(n) \dot{-} 1}\} .$$

Let f witness that C is creative. Define a recursive $\tilde{f}$ by

$$\begin{aligned} \tilde{f}(e, 0) &= \prec f(e) \succ ; \\ \tilde{f}(e, k+1) &= \tilde{f}(e, k) * \prec f(p(e, \tilde{f}(e, k))) \succ . \end{aligned}$$

By induction, we show that, for all k,

(i) $\tilde{f}(e, k) \in \mathrm{Seq}$;

(ii) $\mathrm{lh}(\tilde{f}(e, k)) = k + 1$;

(iii) $(\forall k' \leq k)\, \tilde{f}(e,k') = \tilde{f}(e,k) \lceil k'+1\,;$

(iv) $W_e \cap C = \emptyset \;\rightarrow\; (\forall i \leq k)(\forall j < i)\, (\tilde{f}(e,k))_i \neq (\tilde{f}(e,k))_j\,;$

(v) $W_e \cap C = \emptyset \;\rightarrow\; (\forall i \leq k)\, (\tilde{f}(e,k))_i \notin W_e \cup C\,.$

Clauses (i)–(iii) are clear. To verify (iv) and (v), note that

$$W_{p(e,\tilde{f}(e,k))} = W_e \cup \{(\tilde{f}(e,k))_i \mid i \leq k\}\,.$$

Define h by recursion as follows. If the numbers $(\tilde{f}(e,e))_k,\ k \leq e$, are distinct, let $h(e)$ be the least of these numbers that is different from all the $h(e'),\ e' < e$. Otherwise let $h(e)$ be the least number that is different from all the $h(e'),\ e' < e$. The recursive function h witnesses that C is creative. □

For subsets A and B of ω, say that A is *one-one reducible* to B $(A \leq_1 B)$ if some one-one f witnesses that $A \leq_m B$. Define the notion of a *one-one complete r.e. set* in the obvious way. All our earlier results go through with "one-one" replacing "many-one." Hence we have the following theorem.

Theorem 5.18. *An r.e. set C is creative if and only if C is many-one complete if and only if C is one-one complete.*

A *recursive permutation* is a recursive one-one onto $f : \omega \rightarrow \omega$. Two subsets of ω are *recursively isomorphic* if one is the image of the other under a recursive permutation.

Theorem 5.19. *Let A and B be arbitrary subsets of ω. If $A \leq_1 B$ and $B \leq_1 A$, then A and B are recursively isomorphic.*

Proof. Suppose that g and h witness that $A \leq_1 B$ and $B \leq_1 A$ respectively.

We define inductively recursive functions $p : \omega \rightarrow \omega,\ r : {}^2\omega \rightarrow \omega$, and $s : {}^2\omega \rightarrow \omega$. There will be numbers $m_i\ \ i \in \omega$, and $n_i,\ i \in \omega$, such that, for each k,

$$p(k) = \langle\langle m_0, n_0\rangle, \ldots, \langle m_{2k-1}, n_{2k-1}\rangle\rangle\,.$$

The m_i will be distinct, as will the n_i. Moreover we shall have that

$$m_i \in A \leftrightarrow n_i \in B\,.$$

Given $p(k)$, let m_{2k} be the least number different from all the $m_i,\ i < 2k$. Set $r(k,0) = g(m_{2k})$ and

$$r(k,i+1) = \begin{cases} r(k,i) & \text{if } r(k,i) \notin \{n_0,\ldots,n_{2k-1}\}\,; \\ g(m_j),\ \text{where } n_j = r(k,i), & \text{otherwise.} \end{cases}$$

Since g is one-one, it follows that, whenever $r(k, i+1)$ is defined by the second clause, the numbers $r(k,0)\ldots, r(k,i+1)$ are distinct. For any i, $r(k,i) \in B$ if and only if $m_{2k} \in A$.

Let $n_{2k} = r(k,i)$ for the least $i \leq 2k$ such that $r(k,i) \notin \{n_0, \ldots, n_{2k-1}\}$.

Now let n_{2k+1} be the least number different from all the n_i, $i \leq 2k$. Define $s(k,i)$ and m_{2k+1} by analogy with the definition of $r(k,i)$ and n_{2k}.

Now define $f : \omega \to \omega$ by setting $f(m_i) = n_i$ for each $i \in \omega$. Clearly f witnesses that A and B are recursively isomorphic. $\square$

Corollary 5.20. *Any two creative sets are recursively isomorphic.*

We now turn to the topic of relative recursion. If $f : \omega \to \omega$, then the functions *recursive in* f form the smallest set $\mathcal{C}$ such that

(I) The function S, all constant functions, all the I_i^m, and f belong to $\mathcal{C}$;

(II) $\mathcal{C}$ is closed under composition;

(III) $\mathcal{C}$ is closed under primitive recursion;

(IV) $\mathcal{C}$ is closed under the μ operator.

For $R \subseteq {}^n\omega$, R is *recursive in* f if K_R is recursive in f. The partial functions *partial recursive in* f and the subsets of ω *recursively enumerable in* f are defined in the obvious way.

Let $\mathcal{L}^{\mathrm{PAF}}$ be the result of adding to $\mathcal{L}^{\mathrm{PA}}$ a new one-place function symbol $\mathbf{F}$. For any $f : \omega \to \omega$, let $\mathrm{Q}(f)$ be the set of all consequences (in $\mathcal{L}^{\mathrm{PAF}}$) of Axioms (1)–(8) plus

$$\{\mathbf{F}(\mathbf{S}^a(\mathbf{0})) = \mathbf{S}^{f(a)}(\mathbf{0}) \mid a \in \omega\}\,.$$

Theorem 5.21. *For all f, the functions recursive in f are the same as the functions representable in* $\mathrm{Q}(f)$.

Proof. Our proofs of Theorems 4.17 and 4.33 are easily adapted to give a proof the present theorem, since f is representable in $\mathrm{Q}(f)$ and since the relation Pr for $\mathrm{Q}(f)$ is recurive in f. $\square$

For $n \geq 1$, let T_n^f be defined just as was T_n, but using $\mathrm{Q}(f)$ instead of Q.

Theorem 5.22. *For any f, T_n^f is recursive in f. The functions partial recursive in f are exactly the $\{e\}_n^f$, where*

$$\{e\}_n^f(a_1, \ldots, a_n) \simeq U(\mu d\, T_n^f(e, a_1, \ldots, a_n, d))\,.$$

Whenever $T_n^f(e, a_1, \ldots, a_n, d)$ holds, then $(d)_1$ is the # of some deduction. Any axiom of $\mathrm{Q}(f)$ that occurs as a line in this deduction must have # smaller than d. Hence, for any such axiom of the form $\mathbf{F}(\mathbf{S}^a(\mathbf{0})) = \mathbf{S}^{f(a)}(\mathbf{0})$, we must have that $a < d$. In particular, this means that whether $T_n^f(e, a_1, \ldots, a_n, d)$ holds depends only upon $f \restriction d$.

Define $T_n^1 \subseteq {}^{n+3}\omega$ by letting $T_n^1(c, e, a_1, \ldots, a_n, d)$ hold if and only if

$$c \in \mathrm{Seq} \ \wedge \ \mathrm{lh}(c) = d \ \wedge \ (\forall f)((\forall i < d)\, f(i) = (c)_i \rightarrow T_n^f(e, a_1, \ldots, a_n, d)) .$$

Note that we could have written the definition of T_n^1 directly, without mentioning the f's or the T_n^f's.

Theorem 5.23. *For each n, the relation T_n^1 is primitive recursive. For any f, n, e, and $a_1, \ldots, a_n$,*

$$\{e\}_n^f(a_1, \ldots, a_n) \simeq U(\mu d\, T_n^1(\bar{f}(d), e, a_1, \ldots, a_n, d)) .$$

Let us extend the definition of recursive enumerability to subsets of ${}^n\omega$ by declaring $A \subseteq {}^n\omega$ to be *recursively enumerable* if A is the domain of a partial recursive function. Similarly define the notion of A's being *recursively enumerable in* f, for $f : \omega \rightarrow \omega$ and $A \subseteq {}^n\omega$.

If $n \geq 1$, $A \subseteq {}^n\omega$, and $k \geq 1$, then $A \in \Sigma_k$ (or A *is* Σ_k) if there is a recursive $B \subseteq {}^{n+k}\omega$ such that, for all $a_1, \ldots, a_n$,

$$(a_1, \ldots, a_n) \in A \ \leftrightarrow \ (\exists b_1) \cdots (\mathrm{Q} b_k)\, (a_1, \ldots, a_n, b_1, \ldots, b_k) \in B ,$$

where the quantifiers alternate between $\exists$ and $\forall$ (so that Q is $\exists$ just in case k is odd). Let $A \in \Pi_k$ if and only if $\neg A \in \Sigma_k$. Let $\Delta_k = \Sigma_k \cap \Pi_k$. Similarly define $\Sigma_k(f)$, $\Pi_k(f)$, and $\Delta_k(f)$, replacing the condition that B is recursive with the condition that it is recursive in f. We shall sometimes say, e.g., that A *is* Σ_k *in* f to mean that $A \in \Sigma_k(f)$.

We omit the easy proof of the following theorem.

Theorem 5.24. *Let $n \geq 1$ and $A \subseteq {}^n\omega$. Then A is Σ_1 if and only if A is r.e., and A is Δ_1 if and only if A is recursive. For $f : \omega \rightarrow \omega$, A is Σ_1 in f if and only if A is r.e. in f, and A is Δ_1 in f if and only if A is recursive in f.*

For $f : \omega \rightarrow \omega$, let

$$\begin{aligned} \mathcal{K}^f &= \{e \mid \{e\}_1^f(e) \text{ is defined}\} \\ &= \{e \mid (\exists d)\, T_1^1(\bar{f}(d), e, e, d)\} \\ &= \{e \mid e \in W_e^f\} , \end{aligned}$$

where $W_e^f = \mathrm{domain}\,(\{e\}_1^f)$.

Theorem 5.25. *For all $f : \omega \to \omega$, we have:*

(1) $\mathcal{K}^f$ is r.e. in f ;

(2) $\mathcal{K}^f$ is not recursive in f ;

(3) if $A \subseteq \omega$ is r.e. in f, then $A \leq_1 \mathcal{K}^f$;

(4) f is recursive in $K_{\mathcal{K}^f}$.

Proof. The proofs of (1) and (2) are like the proofs of the corresponding facts for $\mathcal{K}$.

Note that, for each m and n, a definition like that of the S^m_n function gives a one-one recursive function $\tilde{S}^m_n$ such that, for all f, e, $a_1, \ldots, a_m$, and $b_1, \ldots, b_n$,

$$\{\tilde{S}^m_n(e, a_1, \ldots, a_m)\}^f_n(b_1, \ldots, b_n) \simeq \{e\}^f_{m+n}(a_1, \ldots, a_m, b_1, \ldots, b_n)\,.$$

We leave as an exercise the task of using $\tilde{S}^m_n$ to prove (3) and (4). $\square$

Exercise 5.6. Prove parts (3) and (4) of Theorem 5.25.

For $k \in \omega$, define $0^{(k)} : \omega \to \omega$ as follows:

$$\begin{aligned} 0^{(0)} &= K_\emptyset\,; \\ 0^{(k+1)} &= K_{\mathcal{K}^{0^{(k)}}}\,. \end{aligned}$$

Theorem 5.26. *For any $A \subseteq {}^n\omega$, A is Σ_{k+1} if and only if A is r.e. in $0^{(k)}$.*

Proof. The case $k = 0$ follows from Theorems 5.24 and 5.25, so assume that $k \geq 0$ and that the theorem holds for k.

First suppose that A is r.e. in $0^{(k+1)}$. Let e be a number such that $A = \text{domain}\,(\{e\}^{0^{(k+1)}}_n)$. Then, for all $a_1, \ldots, a_n$,

$$\begin{aligned} (a_1, \ldots, a_n) \in A \;&\leftrightarrow\; (\exists d)\, T^1_n(\overline{0^{(k+1)}}(d), e, a_1, \ldots, a_n, d) \\ &\leftrightarrow\; (\exists d)(\exists c)(c = \overline{0^{(k+1)}}(d) \;\wedge\; T^1_n(c, e, a_1, \ldots, a_n, d))\,. \end{aligned}$$

Now

$$c = \overline{0^{(k+1)}}(d) \;\leftrightarrow\; (c \in \text{Seq} \wedge \text{lh}(c) = d \wedge (\forall i < d)\, (c)_i = 0^{(k+1)}(i))\,.$$

Moreover

$$(c)_i = 0^{(k+1)}(i) \;\leftrightarrow\; (((c)_i = 1 \;\wedge\; i \in \mathcal{K}^{0^{(k)}}) \;\vee\; ((c)_i = 0 \;\wedge\; i \notin \mathcal{K}^{0^{(k)}}))\,.$$

Since $\mathcal{K}^{0^{(k)}}$ is r.e. in $0^{(k)}$, we have by induction that $\mathcal{K}^{0^{(k)}}$ is Σ_{k+1}. Thus there is a recursive B such that, for each $i \in \omega$,

$$\begin{aligned} i \in \mathcal{K}^{0^{(k)}} &\leftrightarrow (\exists b_1) \cdots (Q b_{k+1})\,(i, b_1, \ldots, b_{k+1}) \in B\,; \\ i \notin \mathcal{K}^{0^{(k)}} &\leftrightarrow (\forall b_1') \cdots (Q' b_{k+1}')\,(i, b_1', \ldots, b_{k+1}') \notin B\,. \end{aligned}$$

Substituting and bringing all quantifiers to the front, we get that, for all $a_1, \ldots, a_n$, $(a_1, \ldots, a_n) \in A$ if and only if

$$\begin{aligned} &(\exists d)(\exists c)(\forall i < d)(\exists b_1)(\forall b_1') \cdots (Q b_{k+1})(Q' b_{k+1}') \\ &\qquad R(a_1, \ldots, a_n, d, c, i, b_1, b_1', \ldots, b_{k+1}, b_{k+1}')\,, \end{aligned}$$

with R recursive. Now, for any relation P,

$$\begin{aligned} (\forall i < d)(\exists b)\, P(i, b) &\leftrightarrow (\exists \hat{b})(\forall i < d)\, P(i, (\hat{b})_i)\,; \\ (\forall i < d)(\forall b)\, P(i, b) &\leftrightarrow (\forall b)(\forall i < d)\, P(i, b)\,. \end{aligned}$$

Hence we can move $(\forall i < d)$ to the right past all the other quantifiers. Since

$$\begin{aligned} (\exists b)(\exists b')\, P(b, b') &\leftrightarrow (\exists \hat{b})\, P((\hat{b})_0, (\hat{b})_1)\,; \\ (\forall b)(\forall b')\, P(b, b') &\leftrightarrow (\forall \hat{b})\, P((\hat{b})_0, (\hat{b})_1)\,; \end{aligned}$$

we can contract adjacent pairs of like quantifiers. The end result is that we show A to be Σ_{k+2}.

Now suppose that A is Σ_{k+2}. There is then a $C \in \Pi_{k+1}$ such that, for all $a_1, \ldots, a_n$,

$$(a_1, \ldots, a_n) \in A \leftrightarrow (\exists b)(a_1, \ldots, a_n, b) \in C\,.$$

By induction, $\neg C$ is r.e. in $0^{(k)}$. Let

$$D = \{\langle a_1, \ldots, a_n, b \rangle \mid (a_1, \ldots, a_n, b) \notin C\}\,.$$

Then D is r.e. in $0^{(k)}$, and so Theorem 5.25 implies that $D \leq_1 \mathcal{K}^{0^{(k)}}$. By the definition of $0^{(k+1)}$, this gives that D is recursive in $0^{(k+1)}$. But A is Σ_1 in K_D, hence r.e. in K_D, hence r.e. in $0^{(k+1)}$. $\square$

Theorem 5.27. *For each $k \geq 1$,*

$$\Delta_k \subsetneq \Sigma_k \ \wedge\ \Delta_k \subsetneq \Pi_k \ \wedge\ (\Sigma_k \cup \Pi_k) \subsetneq \Delta_{k+1}\,.$$

Proof. That $\Delta_k \subseteq \Sigma_k$ and $\Delta_k \subseteq \Pi_k$ is by definition. Using vacuous quantifiers, we can see that $\Sigma_k \subseteq \Delta_{k+1}$ and $\Pi_k \subseteq \Delta_{k+1}$.

Since $\mathcal{K}^{0^{(k-1)}}$ is r.e. in $0^{(k-1)}$ but not recursive in $0^{(k-1)}$, we have an example of a set that belongs to $\Sigma_k \setminus \Delta_k$. But then $\neg\mathcal{K}^{0^{(k-1)}}$ belongs to $\Pi_k \setminus \Delta_k$.

The join of $\mathcal{K}^{0^{(k-1)}}$ and $\neg\mathcal{K}^{0^{(k-1)}}$ is recursive in $0^{(k)}$ and so belongs to Δ_{k+1}, but it does not belong to $\Sigma_k \cup \Pi_k$. □

For $n \geq 1$, a subset A of ω is *one-one complete for* Σ_n if $A \in \Sigma_n$ and every Σ_n subset of ω is one-one reducible to A. Similarly define *one-one complete for* Π_n, *many-one complete for* Σ_n, and *many-one complete for* Π_n.

Theorem 5.28. *Let A be the set of all $e \in \omega$ such that W_e is finite. Then A is one-one complete for Σ_2.*

Proof. For each $e \in \omega$,

$$\begin{aligned} e \in A &\leftrightarrow (\exists m)(\forall n)(n \in W_e \to n \leq m) \\ &\leftrightarrow (\exists m)(\forall n)(\forall d)(T_1(e,n,d) \to n \leq m)\,. \end{aligned}$$

Thus $A \in \Sigma_2$.

Let $B \subseteq \omega$ with $B \in \Sigma_2$. There is a recursive C such that

$$(\forall e)(e \in B \leftrightarrow (\exists m)(\forall n)\,(e,m,n) \in C)\,.$$

Define $f : {}^2\omega \to \omega$ by

$$f(e,m) \simeq \mu n\,(\forall m' \leq m)(\exists n' \leq n)\,(e,m',n') \notin C\,.$$

Since f is partial recursive, the *s-m-n* Theorem gives us a one-one recursive g such that

$$(\forall e)(\forall m)\,\{g(e)\}(m) \simeq f(e,m)\,.$$

To see that g witnesses that $B \leq_1 A$, assume first that $e \in B$. Then there is an m such that $(e,m,n) \in C$ for all n. For $m' \geq m$, $f(e,m')$ is undefined. Hence $W_{g(e)} \subseteq m$.

Now assume that $e \notin B$. Then for every m there is an n such that $(e,m,n) \notin C$. Thus $f(e,m)$ is defined for every m, and so $W_{g(e)} = \omega$. □

Exercise 5.4 gives an example of a set many-one complete (indeed, one-one complete) for Σ_1.

Exercise 5.7. Show that $\{e \mid \text{range}(\{e\}) = \omega\}$ is one-one complete for Π_2.

Exercise 5.8. Show that $\{e \mid \neg W_e \text{ is finite}\}$ is one-one complete for Σ_3.

Degrees of unsolvability.

For $f : \omega \to \omega$, define the *degree* $\boldsymbol{d}(f)$ of f by

$$\boldsymbol{d}(f) = \{g \in {}^{\omega}\omega \mid f \leq_T g \wedge g \leq_T f\},$$

where $\leq_T$ means "is recursive in." Let

$$\mathcal{D} = \{\boldsymbol{d}(f) \mid f \in {}^{\omega}\omega\}.$$

$\mathcal{D}$ is the set of *degrees of unsolvability.* Partially order $\mathcal{D}$ by

$$\boldsymbol{d}(f) \leq \boldsymbol{d}(g) \leftrightarrow f \leq_T g.$$

Theorem 5.29. *The structure $(\mathcal{D}, \leq)$ is an upper semilattice with a least element.*

Proof. The least upper bound of degrees $\boldsymbol{d}(f_1)$ and $\boldsymbol{d}(f_2)$ is $\boldsymbol{f}$, where for each n,

$$\begin{aligned} f(2n) &= f_1(n); \\ f(2n+1) &= f_2(n). \end{aligned}$$

The recursive functions all have the same degree $\mathbf{0}$, and this is the least degree. □

Theorem 5.30. *There exist incomparable degrees, i.e., $\leq$ is not a linear ordering of $\mathcal{D}$.*

Proof. We define inductively $s_0, s_1, \ldots$ and $t_0, t_1, \ldots$ such that

(a) $(\forall i \in \omega)\, s_i \in \text{Seq}$;

(b) $(\forall i \in \omega)\, t_i \in \text{Seq}$;

(c) $(\forall i \in \omega)(\forall j \in \omega)(i < j \to (\text{lh}(s_i) < \text{lh}(s_j) \wedge s_i = s_j \lceil \text{lh}(s_i)))$;

(d) $(\forall i \in \omega)(\forall j \in \omega)(i < j \to (\text{lh}(t_i) < \text{lh}(t_j) \wedge t_i = t_j \lceil \text{lh}(t_i)))$.

Let $s_0 = t_0 = \langle\ \rangle$.

Assume that s_e and t_e are defined.

If there is an $s \in \mathrm{Seq}$ such that

(i) $\mathrm{lh}(s) \geq \mathrm{lh}(s_e)$;

(ii) $s \restriction \mathrm{lh}(s_e) = s_e$;

(iii) $(\exists d \leq \mathrm{lh}(s))\, T_1^1(s\lceil d, e, \mathrm{lh}(t_e), d)$;

then let s'_e be the least such s and let

$$t'_e = t_e * \langle U(\mu d\, T_1^1(s'_e\lceil d, e, \mathrm{lh}(t_e), d)) + 1\rangle\,.$$

Otherwise let $s'_e = s_e$ and $t'_e = t_e$.

If there is a $t \in \mathrm{Seq}$ such that

(i) $\mathrm{lh}(t) > \mathrm{lh}(t'_e)$;

(ii) $t \restriction \mathrm{lh}(t'_e) = t'_e$;

(iii) $(\exists d \leq \mathrm{lh}(t))\, T_1^1(t\lceil d, e, \mathrm{lh}(s'_e), d)$;

then let t_{e+1} be the least such t and let

$$s_{e+1} = s'_e * \langle U(\mu d\, T_1^1(t_{e+1}\lceil d, e, \mathrm{lh}(s'_e), d)) + 1\rangle\,.$$

Otherwise let $t_{e+1} = t'_e * \langle 0\rangle$ and let $s_{e+1} = s'_e * \langle 0\rangle$.

Let $f : \omega \to \omega$ be such that $\bar{f}(\mathrm{lh}(s_i)) = s_i$ for all $i \in \omega$ and let $g : \omega \to \omega$ be such that $\bar{g}(\mathrm{lh}(t_i)) = t_i$ for all $i \in \omega$.

To show that $g \not\leq_T f$, let $e \in \omega$. We show that $\{e\}^f \neq g$. To see this, note that $\{e\}^f(\mathrm{lh}(t_e)) \neq g(\mathrm{lh}(t_e))$; for, if $\{e\}^f(\mathrm{lh}(t_e))$ is defined, then

$$g(\mathrm{lh}(t_e)) = t'_e(\mathrm{lh}(t_e)) = \{e\}^f(\mathrm{lh}(t_e)) + 1\,.$$

Similarly, for each $e \in \omega$,

$$f(\mathrm{lh}(s'_e)) = s_{e+1}(\mathrm{lh}(s'_e)) = \{e\}^g(\mathrm{lh}(s'_e)) + 1\,.$$

Hence $f \not\leq_T g$. $\square$

For subsets A of ω, let $\boldsymbol{d}(A) = \boldsymbol{d}(K_A)$. A degree is *recursively enumerable* if it is $\boldsymbol{d}(A)$ for some r.e. A. There is a least r.e. degree, $\mathbf{0}$, and there is a greatest r.e. degree, $\mathbf{0'} = \boldsymbol{d}(0^{(1)}) = \boldsymbol{d}(\mathcal{K})$.

Theorem 5.31. *There is an r.e. degree* $\boldsymbol{d}$ *such that*

$$\mathbf{0} < \boldsymbol{d} < \mathbf{0'}\,.$$

Proof. We shall construct a recursive function $f : {}^2\omega \to \omega$ satisfying

$$(\forall s)(\forall e)(f(s,e) > 0 \to (e < s \wedge f(s+1,e) = f(s,e)))\,.$$

For $s \in \omega$, we let

$$A^s = \{f(s,e) - 1 \mid f(s,e) > 0\}\,.$$

The stated properties of f imply the recursiveness of $\{(s,m) \mid m \in A^s\}$. For $e \in \omega$, we let

$$m_e \simeq f(\mu s\, f(s,e) > 0, e) - 1\,.$$

Finally we let

$$A = \{m_e \mid m_e \text{ is defined}\} = \bigcup_s A^s\,.$$

Thus A will be r.e.

We shall make $\boldsymbol{d}(A) > \mathbf{0}$ by arranging that $\neg A$ is infinite and, for all e,

(I) $$W_e \text{ is infinite } \to\ W_e \cap A \neq 0\,.$$

The numbers m_e will be used to witness that (1) holds.

We shall make $\boldsymbol{d}(A) < \mathbf{0'}$ by arranging that

(II) $$\mathcal{K}^{K_A} \in \Delta_2\,.$$

By Theorem 5.26, (2) implies that $\mathcal{K}^{K_A} \leq_T 0^{(1)}$, and so that

$$\boldsymbol{d}(A) < \boldsymbol{d}(\mathcal{K}^{K_A}) \leq \mathbf{0'}\,.$$

As we define f, we shall simultaneously define another recursive function $g : {}^2\omega \to \omega$.

Set $f(0,e) = 0$ for all e.

Let $s \in \omega$. Suppose $f(s,e)$ is defined for all e. Suppose inductively that

$$(\forall e)(f(s,e) > 0 \to W_e^s \cap A^s \neq \emptyset)\,,$$

where

$$W_e^s = \{n \mid (\exists d \leq s)\, T_1(e,n,d)\}\,.$$

For each e, let

$$g(s,e) = \begin{cases} \mu d\,(d \leq s \wedge T_1^1(\overline{K_{A^s}}(d), e, e, d)) & \text{if } (\exists d \leq s)\, T_1^1(\overline{K_{A^s}}(d), e, e, d)\,; \\ 0 & \text{otherwise.} \end{cases}$$

For each $e < s$, if both

(a) $W_e \cap A^s = \emptyset$;

(b) $(\exists m \leq s)(m \in W_e^s \wedge m > 2e \wedge (\forall e' < e)\, m \geq g(s, e'))$;

then, for the least such m, let $f(s+1, e) = m + 1$. If either (a) or (b) does not hold, let $f(s+1, e) = f(s, e)$.

Lemma 5.32. *$\neg A$ is infinite.*

Proof. Each $m_e > 2e$, and therefore

$$\{n \mid n \in A \wedge n \leq 2e\} \subseteq \{m_{e'} \mid e' < e\},$$

a set of size $\leq e$. $\square$

Lemma 5.33. *For each e, $\lim_s g(s, e)$ exists.*

Proof. Fix e. Let s_0 be such that

$$(\forall e' \leq e)\,(m_e \text{ defined} \rightarrow f(s_0, e') > 0)\,.$$

Suppose that $s \geq s_0$ and $g(s, e) > 0$. Any e' such that $f(s, e') = 0$ and $f(s+1, e') > 0$ must be greater than e, and so, by condition (b) above, must satisfy $m_{e'} \geq g(s, e)$. Thus $A^{s+1} \cap g(s, e) = A^s \cap g(s, e)$. This implies that $g(s+1, e) = g(s, e)$. We have then shown that if $g(s, e) > 0$ for some $s \geq s_0$ then $g(s', e) = g(s, e)$ for every $s' \geq s$. $\square$

Lemma 5.34. *For each e, $\lim_s g(s, e) > 0$ if and only if $e \in \mathcal{K}^{K_A}$.*

Proof. Let $\hat{g}(e) = \lim_s g(s, e)$ and assume that $\hat{g}(e) > 0$. for all sufficiently large s, $A^s \cap \hat{g}(e) = A \cap \hat{g}(e)$. By the definition of $g(s, e)$, $e \in \mathcal{K}^{K_A}$.

Now assume that $e \in \mathcal{K}^{K_A}$. Then $(\exists d)\, T_1^1(\overline{K_A}(d), e, e, d)$. Hence, for every large enough s, $(\exists d \leq s)\, T_1^1(\overline{K_{A^s}}(d), e, e, d)$, and so $g(s, e) > 0$. $\square$

Lemma 5.35. *(1) holds.*

Proof. Let $e \in \omega$ and suppose that W_e is infinite. Let $m \in W_e$ with $m > 2e$ and $m \geq \hat{g}(e')$ for all $e' < e$. Let s be such that $e < s$, $m \leq s$, $m \in W_e^s$, and $g(s, e') = \hat{g}(e')$ for all $e' < e$. If $W^s \cap A^s = \emptyset$, then (a) and (b) hold for m at s, and so some $m' \leq m$ belongs to $W_e^s \cap A^{s+1}$. $\square$

Lemma 5.36. *(2) holds.*

Proof. For each e,

$$
\begin{aligned}
e \in \mathcal{K}^{K_A} &\leftrightarrow \lim_s g(s,e) > 0 \\
&\leftrightarrow (\exists s)(\forall s')(s' \geq s \rightarrow g(s',e) > 0) \\
&\leftrightarrow (\forall s)(\exists s')(s' \geq s \wedge g(s',e) > 0)\,.
\end{aligned}
$$

$\square$

Exercise 5.9. Prove that there is set of size $2^{\aleph_0}$ of pairwise incomparable degrees of unsolvability.

Hint. Modify the proof of Theorem 5.30 by defining $\langle s_u \mid u \in {}^{<\omega}2\rangle$.

Exercise 5.10. Show that there is no partial recursive function f such that, for all $e \in \omega$, if $\neg W_e$ is finite then $f(e)$ is defined and every number $\geq f(e)$ belongs to W_e.

Exercise 5.11. Show that there are recursive functions $f : {}^2\omega \rightarrow \omega$ and $g : {}^2\omega \rightarrow \omega$ such that

(a) for all e_1 and e_2, $W_{f(e_1,e_2)}$ and $W_{g(e_1,e_2)}$ are disjoint and recursive;

(b) for all e_1 and e_2, if $W_{e_1} = \neg W_{e_2}$ then $W_{f(e_1,e_2)} = W_{e_1}$ and $W_{g(e_1,e_2)} = W_{e_2}$.

Hint. All finite sets are recursive.

Exercise 5.12. Let A be a recursively enumerable set such that $\neg A$ is infinite. Let $f : \omega \rightarrow \neg A$ be one-one onto and order preserving. Assume that f eventually dominates every partial recursive function, i.e., that, for every partial recursive g,

$$(\exists m)(\forall n \geq m)(g(n) \text{ is defined} \rightarrow g(n) \leq f(n))\,.$$

Prove that $\boldsymbol{d}(A) = \mathbf{0}'$.

6 Constructible Sets

In this section, as in §1, we we our notation and terminology is pretty much the same as that of Kenneth Kunen's *Set Theory: an Introduction to Independence Proofs*. In addition, our treatment of constructible sets is derived from Kunen's.

In ZFC without the axiom of Foundation, we proved (Theorem 1.9) the existence of the class function $\alpha \mapsto V_\alpha$. Still working in ZFC − Foundation, we can define the proper class WF by

$$\mathrm{WF} = \bigcup\{V_\alpha \mid \alpha \in \mathrm{ON}\}\,.$$

Moreover it is easy to convince oneself that all the axioms of ZFC, including Foundation, hold in $(\mathrm{WF}; \in \restriction \mathrm{WF})$. Can one not show in this way the consistency of the Axiom of Foundation? The answer is yes, but we have to be careful about several things.

We can't hope to show that the consistency of ZFC is a theorem of ZFC − Foundation, for the second incompleteness theorem implies that the consistency of ZFC cannot be proved even in ZFC (unless ZFC is inconsistent). Of course, the argument outlined above doesn't actually establish the consistency of ZFC, since $(\mathrm{WF}; \in \restriction \mathrm{WF})$ isn't actually a (set) model. And we can't really be "working in ZFC − Foundation" if we show that all the axioms of ZFC hold in WF, for this assertion isn't even expressible in the formal language of set theory.

Let M be a class. For formulas φ (of the language of set theory), we define φ^M, the *relativization* of φ to M, inductively as follows:

(a) $(x = y)^M$ is $x = y$;

(b) $(x \in y)^M$ is $x \in y$;

(c) $(\neg\varphi)^M$ is $\neg\varphi^M$;

(d) $(\varphi \wedge \psi)^M$ is $(\varphi^M \wedge \psi^M)$;

(e) $((\exists x)\varphi)^M$ is $(\exists x)(x \in M \,\wedge\, \varphi^M)$.

This definition requires some explanation.

Classes are the (sometimes nonexistent, from the point of view of ZFC) extensions of formulas. So we should think of M as being $\{x \mid \chi(x)\}$ for some formula χ. Thus clause (e) should really read

(e) $((\exists x)\,\varphi)^M$ is $(\exists x)(\chi(x) \,\wedge\, \varphi^M)$.

Hence the operation $\varphi \mapsto \varphi^M$ depends not just on M but also on a formula χ defining M.

Even this amended account of the definition is not really accurate. A class need not be *definable*. It may be given by a formula $\chi(x, y_1, \ldots, y_n)$. (If we are *using* the language, then the formula is, in effect, specifying for us a class; if we are *talking about* the language, then the formula isn't specifying a class unless we assign sets to the variables y_i.) For classes M given by such formulas, the definition of φ^M must be modified so that the quantifiers of φ^M do not bind any of the variables $y_1, \ldots, y_n$ occurring free in the the defining formula.

For any class M and formula φ, *φ is true in M*, *φ holds in M*, and *M is a model of φ* all mean the same as the formula φ^M.

Lemma 6.1. *Let S and T be sets of sentences in the language of set theory and let M be a definable class. Suppose that (for some formula defining M)*

(1) $T \models M \neq \emptyset$;

(2) $(\forall \sigma \in S)\, T \models \sigma^M$.

Then S is consistent if T is consistent.

Proof. Let $\chi(x)$ be the given formula defining M for which (1) and (2) hold. (Note that the Lemma is really about χ and has nothing to do with M *qua* class.)

Assume that T is consistent. Let $\mathfrak{A}$ be a model of T. Let $\mathfrak{B}$ be given by

$$\begin{aligned} B &= \{a \in A \mid \mathfrak{A} \models \chi[a]\}\,; \\ \in_{\mathfrak{B}} &= \in_{\mathfrak{A}} \restriction B\,. \end{aligned}$$

(1) implies that $B \neq \emptyset$ and so that $\mathfrak{B}$ is a model. It is routine to show that, for any sentence σ,

$$\mathfrak{B} \models \sigma \;\leftrightarrow\; \mathfrak{A} \models \sigma^M\,.$$

Thus (2) implies that $\mathfrak{B} \models S$. □

Remarks:

(a) It is easy to give a direct proof-theoretic argument for the (equivalent) version of Lemma 6.1 formulated in terms of deductive consistency.

(b) Suppose that S and T are, say, recursively axiomatizable theories. Then the deductive consistency version of Lemma 6.1 for S and T can be formulated in, for example, Peano Arithmetic. Moreover it can be proved in PA. The applications we make of Lemma 6.1 will all involve recursively axiomatizable theories, and the arithemetic versions of (1) and (2) will be provable in PA. Thus our relative consistency results are all essentially theorems of PA.

Lemma 6.2. *If M is a transitive class, then the Axiom of Extensionality holds in M.*

Proof. Let M be transitive. The relativization of Extensionality to M is equivalent to

$$(\forall x \in M)(\forall y \in M)((\forall z \in M)(z \in x \leftrightarrow z \in y) \rightarrow x = y)\,.$$

Fix elements x and y of M and assume that $(\forall z \in M)(z \in x \leftrightarrow z \in y)$. Since M is transitive, this implies that $(\forall z)(z \in x \leftrightarrow z \in y)$. By Extensionality, $x = y$. $\square$

Lemma 6.3. *The Axiom of Foundation holds in every subclass of* WF.

Proof. Let $M \subseteq \mathrm{WF}$. The relativization of Foundation to M is

$$(\forall x \in M)((\exists y \in M)\, y \in x \;\rightarrow\; (\exists y \in x \cap M)(\forall z \in x \cap M)\, z \notin y)\,.$$

Let $x \in M$. Assume that $x \cap M \neq \emptyset$. Since $M \subseteq \mathrm{WF}$, there is a least ordinal α such that $x \cap M \cap V_\alpha \neq \emptyset$. For this least α, let $y \in x \cap M \cap V_\alpha$. Since all members of y belong to V_β for some $\beta < \alpha$, y is disjoint from $x \cap M$. $\square$

Lemma 6.4. *Let M be a class with the following property: For each formula $\varphi(x, z, w_1, \ldots, w_n)$ and for any elements z, w_1, …, w_n of M,*

$$\{x \in z \mid \varphi^M(x, z, w_1, \ldots, w_n)\} \in M\,.$$

Then every instance of the Axiom Schema of Comprehension holds in M.

Proof. Any relativization to M of an instance of Comprehension is of the form

$$(\forall w_1 \in M) \cdots (\forall w_n \in M)(\forall z \in M)(\exists y \in M)(\forall x \in M)(x \in y \leftrightarrow (x \in z \wedge \varphi^M))\,,$$

for φ as in the statement of the lemma. Fix such a φ and fix elements z, $w_1, \ldots w_n$ of M. Let $y = \{x \in z \mid \varphi^M(x, z, w_1, \ldots, w_n)\}$. By hypothesis, $y \in M$. Since $(\forall x)(x \in y \leftrightarrow (x \in z \wedge \varphi^M))$, we have in particular that $(\forall x \in M)(x \in y \leftrightarrow (x \in z \wedge \varphi^M))$. $\square$

In our applications, M will be transitive, so that the set y will not have elements $x \notin M$ satisfying $\varphi^M(x)$. Note that a class M (transitive or not) satisfies the hypothesis of Lemma 6.4 if M if all subsets of elements of M belong to M.

The following two lemmas are easy to prove.

Lemma 6.5. *If M is a class such that, for all x and y belonging to M, there is a $z \in M$ with $\{x, y\} \subseteq z$, then the Axiom of Pairing holds in M.*

Lemma 6.6. *If M is a class such that for all $x \in M$ there is a $y \in M$ such that $\mathcal{U}(x) \subseteq y$, then the Axiom of Union holds in M.*

Lemma 6.7. *Let M be a class with the following property: For each formula $\varphi(x, z, w_1, \ldots, w_n)$ and for any elements z, w_1, $\ldots$, w_n of M, if*

$$(\forall x \in z \cap M)(\exists! y \in M)\varphi^M(x, y, z, w_1, \ldots, w_n)\,,$$

then there is a $u \in M$ such that

$$\{y \in M \mid (\exists x \in z \cap M)\varphi^M(x, y, z, w_1, \ldots, w_n)\} \subseteq u\,.$$

Then every instance of the Axiom Schema of Replacement holds in M.

Proof. The proof is similar to that of Lemma 6.4. □

We postpone discussing the Axioms of Infinity, Power Set, and Choice until we have proved some results about *absoluteness*.

Let $\varphi(x_1, \ldots, x_n)$ be a formula. If M and N are classes such that $M \subseteq N$, then φ is *absolute* for (M, N) if, for any elements $x_1, \ldots x_n$ of M,

$$\varphi^M(x_1, \ldots, x_n) \leftrightarrow \varphi^N(x_1, \ldots, x_n)\,.$$

We say that φ is *absolute* for a class M if φ is absolute for (M, V), i.e., if, for any elements $x_1, \ldots, x_n$ of M,

$$\varphi^M(x_1, \ldots, x_n) \leftrightarrow \varphi(x_1, \ldots, x_n)\,.$$

Lemma 6.8. *If $M \subseteq N$, then the set of formulas absolute for (M, N) is closed under negation and conjunction.*

Proof. The lemma follows directly from the facts that the relativization of $\neg\varphi$ is the negation of the relativization of φ and that the relativization of $\varphi \wedge \psi$ is the conjunction of the relativizations of φ and ψ. □

Lemma 6.9. *Let M and N be transitive classes such that $M \subseteq N$. Then the set of formulas absolute for (M, N) is closed under* bounded quantification*; that is to say, if φ is absolute for (M, N) then*

$$(\exists x)(x \in y \wedge \varphi)$$

is absolute for (M, N).

Proof. Given $\varphi(x, y, z_1, \ldots, z_n)$ absolute for (M, N) and given elements y, $z_1, \ldots, z_n$ of M, we have

$$\begin{aligned} &((\exists x)(x \in y \wedge \varphi(x, y, z_1, \ldots, z_n)))^M \\ &\qquad \leftrightarrow \ (\exists x)(x \in y \wedge \varphi^M(x, y, z_1, \ldots, z_n)) \\ &\qquad \leftrightarrow \ (\exists x)(x \in y \wedge \varphi^N(x, y, z_1, \ldots, z_n)) \\ &\qquad \leftrightarrow \ ((\exists x)(x \in y \wedge \varphi(x, y, z_1, \ldots, z_n)))^N . \end{aligned}$$

The first biconditional follows from the transitivity of M, the second from the absoluteness of φ for (M, N), and the third from the transitivity of N. $\square$

The Δ_0 formulas form the smallest set of formulas satisfying the following conditions:

(1) All atomic formulas are Δ_0.

(2) If φ is Δ_0 then so is $\neg\varphi$.

(3) If φ and ψ are Δ_0 then so is $(\varphi \wedge \psi)$.

(4) If φ is Δ_0 then so is $(\exists x)(x \in y \wedge \varphi)$.

Lemma 6.10. *If M and N are transitive classes and $M \subseteq N$, then all Δ_0 formulas are absolute for (M, N).*

The following useful lemma is easy to prove.

Lemma 6.11. *Let T be a theory and let $\varphi(x_1, \ldots, x_n)$ and $\psi(x_1, \ldots x_n)$ be formulas such that*

$$T \ \models \ (\forall x_1) \cdots (\forall x_n)(\varphi(x_1, \ldots, x_n) \leftrightarrow \psi(x_1, \ldots, x_n)) .$$

Let M and N be models of T such that $M \subseteq N$. Then φ is absolute for (M, N) if ψ is.

If T is a theory in the language of set theory and $\varphi(v_1, \ldots, v_{n+1})$ is a formula of that language, then φ *defines an operation (of n arguments) in* T if

$$T \models (\forall v_1) \cdots (\forall v_n)(\exists! v_{n+1})\varphi(v_1, \ldots, v_{n+1}) .$$

To have a uniform terminology, let us speak of any formula $\varphi(v_1, \ldots, v_n)$ as *defining an n-ary relation in T*. When we speak of a defined operation or relation as being *absolute*, we mean that the defining formula is absolute.

Let ZF be ZFC $-$ Choice.

Lemma 6.12. *The following relations and operations are defined in ZF $-$ Foundation $-$ Power Set $-$ Infinity by formulas provably equivalent in ZF $-$ Foundation $-$ Power Set $-$ Infinity to Δ_0 formulas. Hence they are absolute for any transitive class M that is a model of ZF $-$ Foundation $-$ Power Set $-$ Infinity.*

(a) $x \in y$;	(h) $x \cup y$;
(b) $x = y$;	(i) $x \cap y$;
(c) $x \subseteq y$;	(j) $x \setminus y$;
(d) $\{x, y\}$;	(k) $\mathcal{S}(x)$;
(e) $\{x\}$;	(l) x is transitive ;
(f) $\langle x, y \rangle$;	(m) $\mathcal{U}(x)$;
(g) $\emptyset$;	(n) $\bigcap x$.

In (n), we construe $\bigcap \emptyset$ to be $\emptyset$ in order to make $\bigcap$ into an operation.

Proof. That we defined these relations and functions in ZF $-$ Foundation $-$ Power Set $-$ Infinity, we leave to the reader to check. We content ourselves with making it clear that the defining formulas are equivalent in that theory to Δ_0 formulas.

(a) and (b) are obvious.

For (c), note that $x \subseteq y$ if and only if $(\forall z \in x)\, z \in y$.

For (d), observe that

$$z = \{x, y\} \;\leftrightarrow\; (x \in z \wedge y \in z \wedge (\forall w \in z)(w = x \vee w = y)) .$$

(e) is similar.

For (f), note that $z = \langle x, y \rangle$ if and only if

$$(\exists w \in z)\, w = \{x\} \wedge (\exists w \in z)\, w = \{x, y\} \wedge (\forall w \in z)(w = \{x\} \vee w = \{x, y\}) .$$

Since $w = \{x\}$ and $w = \{x, y\}$ are equivalent to Δ_0 formulas, so is $z = \langle x, y \rangle$.

For (g)–(k), note that

$$\begin{aligned}
z = \emptyset \;&\leftrightarrow\; (\forall w \in z)\, w \neq w\,;\\
z = x \cup y \;&\leftrightarrow\; x \subseteq z \,\wedge\, y \subseteq z \,\wedge\, (\forall w \in z)(w \in x \,\vee\, w \in y)\,;\\
z = x \cap y \;&\leftrightarrow\; z \subseteq x \,\wedge\, z \subseteq y \,\wedge\, (\forall w \in x)(w \in y \rightarrow w \in z)\,;\\
z = x \setminus y \;&\leftrightarrow\; z \subseteq x \,\wedge\, z \cap y = \emptyset \,\wedge\, (\forall w \in x)(w \notin y \rightarrow w \in z)\,;\\
z = \mathcal{S}(x) \;&\leftrightarrow\; x \in z \,\wedge\, x \subseteq z \wedge (\forall w \in z)(w = x \,\vee\, w \in x)\,.
\end{aligned}$$

For (l), observe that x is transitive if and only if $(\forall z \in x)(\forall w \in z)\, w \in x$.
For (m) and (n), note that

$$y = \mathcal{U}(x) \;\leftrightarrow\; (\forall z \in x)\, z \subseteq y \,\wedge\, (\forall z \in y)(\exists w \in x)\, z \in w$$

and that

$$\begin{aligned}
y = \bigcap x \;\leftrightarrow\; &(\forall z \in x)\, y \subseteq z \,\wedge\, (\forall z \in x)(\forall w \in z)((\forall u \in x)\, w \in u \rightarrow w \in y)\\
&\wedge (x = \emptyset \rightarrow y = \emptyset)\,.
\end{aligned}$$

□

Lemma 6.13. *Suppose that M is a transitive model of ZF − Foundation − Power Set − Infinity such that $(\forall x \in M)(\exists y \in M)\, \mathcal{P}(x) \cap M \subseteq y$. Then the Axiom of Power Set holds in M.*

Proof. The relativization to M of Power Set is

$$(\forall x \in M)(\exists y \in M)(\forall z \in M)((z \subseteq x)^M \rightarrow z \in y)\,.$$

By Lemma 6.12, $\subseteq$ is absolute for M, so the relativization of Power Set to M is equivalent to

$$(\forall x \in M)(\exists y \in M)(\forall z \in M)(z \subseteq x \rightarrow z \in y)\,.$$

But this is just what the second part of the hypothesis of the lemma says.

□

Remark. Since $\subseteq$ is literally defined by a Δ_0 formula, the lemma holds without the assumption that M is a model of ZF − Foundation − Power Set − Infinity.

Lemma 6.14. *Let M be a transitive model of ZF − Foundation − Power Set − Infinity. If $\omega \in M$, then the Axiom of Infinity holds in M.*

Proof. The relativization to M of Infinity is

$$(\exists x \in M)(\emptyset^M \in x \wedge (\forall y \in x \cap M)\, \mathcal{S}^M(y) \in x)\,.$$

By the transitivity of M and the absoluteness of $\emptyset$ and $\mathcal{S}$, this is equivalent to

$$(\exists x \in M)(\emptyset \in x \wedge (\forall y \in x)\, \mathcal{S}(y) \in x)\,.$$

But ω witnesses that this is true. $\square$

Lemma 6.15. (Uses Choice) *Let M be a transitive model of ZF − Foundation − Power Set − Infinity such that every subset of an element of M belongs to M. Then the Axiom of Choice holds in M.*

Proof. Using the transitivity of M and the absoluteness of $\emptyset$ and $\cap$, we get that the relativization to M of Choice is

$$\begin{aligned}
&(\forall x \in M)\\
&\quad ((\forall y_1)(\forall y_2)\,((y_1 \in x \wedge y_2 \in x) \to (y_1 \neq \emptyset \wedge (y_1 = y_2 \vee y_1 \cap y_2 = \emptyset)))\\
&\qquad \to\ (\exists z \in M)(\forall y)(y \in x \to (\exists! w \in M)\, w \in y \cap z)))
\end{aligned}$$

Let $x \in M$ satisfy the antecedent of the conditional. Let z be given by Choice. Then

$$(\forall y)(y \in x \to (\exists! w)\, w \in y \cap z)\,.$$

The transitivity of M implies that

$$(\forall y)(y \in x \to (\exists! w \in M)\, w \in y \cap z)\,.$$

This in turn implies that

$$(\forall y)(y \in x \to (\exists! w \in M)\, w \in y \cap (z \cap \mathcal{U}(x))\,.$$

Since the $\mathcal{U}$ operation is defined in M and is absolute for M, the set $\mathcal{U}(x)$ belongs to M. Since $z \cap \mathcal{U}(x) \subseteq \mathcal{U}(x)$, the hypotheses of the lemma give that $z \cap \mathcal{U}(x) \in M$. $\square$

Theorem 6.16. *(a) The class* WF *is a model of ZF.*
(b) (Uses Choice) *The class* WF *is a model of ZFC.*

Proof. Since WF is transitive, Lemma 6.2 implies that Extensionality holds in WF. Since WF $\subseteq$ WF, Lemma 6.3 gives that Foundation holds in WF. All subsets of WF belong to WF, so, by the remark after the proof of Lemma 6.4, Comprehension holds in WF. It is easy to see that WF is closed under the operations of pairing and $\mathcal{U}$; hence Pairing and Union hold in WF by Lemmas 6.5 and 6.6. We leave as an exercise to prove that the hypothesis of Lemma 6.7 holds for WF. By that lemma we then get that Replacement holds in WF. We now have that WF is a model of ZF $-$ Foundation $-$ Power Set $-$ Infinity. For $x \in \mathrm{WF}$,

$$\mathcal{P}(x) \cap \mathrm{WF} = \mathcal{P}(x) \in \mathrm{WF}\,.$$

Hence, by Lemma 6.13, Power Set holds in WF. By Lemma 6.14 and the fact that $\omega \in \mathrm{WF}$, we have that Infinity holds in WF. Since the hypotheses of Lemma 6.15 hold in WF, Choice holds in WF if it holds in V. □

Theorem 6.17. *(a) If ZF $-$ Foundation is consistent, then so is ZF.*
(b) If ZFC $-$ Foundation is consistent, then so is ZFC.

Proof. (a) follows from Lemma 6.1 and and part (a) Theorem 6.16, and (b) follows from Lemma 6.1 and and part (b) Theorem 6.16. □

Exercise 6.1. Prove the the Schema of Replacement holds in WF.

Announcement. We shall no longer note uses of Foundation.

Lemma 6.18. *The composition of absolute operations and relations is absolute: Suppose that T is a theory, that $M \subseteq N$, that M and N are models of T, and that that $G_1, \ldots, G_m$ are n-argument operations defined in T that are absolute for (M, N).*

(a) Let R be an m-ary relation defined in T that is absolute for (M, N). Then the n-ary relation R' given by

$$R'(x_1, \ldots, x_n) \leftrightarrow R(G_1(x_1, \ldots, x_n), \ldots, G_m(x_1, \ldots, x_n))$$

is defined in T and is absolute for (M, N).

(b) Let F be an m-argument operation defined in T that is absolute for (M, N). Then the n-argument operation H given by

$$H(x_1, \ldots, x_n) = F(G_1(x_1, \ldots, x_n), \ldots, G_m(x_1, \ldots, x_n))$$

is defined in T and is absolute for (M, N).

We prove (b). The argument for (a) is similar. It is easy to see that H is defined in T. To prove its absoluteness, let $x_1, \ldots, x_n$ be elements of M. Then

$$\begin{aligned} H^M(x_1, \ldots, x_n) &= F^M(G_1^M(x_1, \ldots, x_n), \ldots, G_m^M(x_1, \ldots, x_n)) \\ &= F^N(G_1^M(x_1, \ldots, x_n), \ldots, G_m^M(x_1, \ldots, x_n)) \\ &= F^N(G_1^N(x_1, \ldots, x_n), \ldots, G_m^N(x_1, \ldots, x_n)) \\ &= H^N(x_1, \ldots, x_n). \end{aligned}$$ □

Lemma 6.19. *The following relations and operations are defined in ZF − Power Set and are absolute for transitive models of ZF − Power Set.*

(a) z *is an ordered pair;*

(b) $u \times v$ *;*

(c) z *is a relation;*

(d) $\mathrm{domain}\,(z)\ \ (= \{x \mid (\exists y)\langle x, y\rangle \in z\})$ *;*

(e) $\mathrm{range}\,(z)\ \ (= \{y \mid (\exists x)\langle x, y\rangle \in z\})$ *;*

(f) z *is a function;*

(g) $z(x)\ \ \left(= \begin{cases} z(x) & \textit{if } (\exists! y)\,\langle x, y\rangle \in z; \\ \emptyset & \textit{otherwise}; \end{cases}\right)$

(h) z *is a one-one function.*

Proof. (a) z is an ordered pair if and only if $(\exists x{\in}\mathcal{U}(z))(\exists y{\in}\mathcal{U}(z))\, z = \langle x, y\rangle$.

(b) The first of our two proofs of the existence of $u \times v$ was in ZF − Power Set, so $\times$ is defined in ZF − Power Set. For absoluteness, note that $z = u \times v$ if and only if

$$(\forall x \in u)(\forall y \in v)\, \langle x, y\rangle \in z \ \wedge\ (\forall w \in z)(\exists x \in u)(\exists y \in v)\, w = \langle x, y\rangle .$$

(c) z is a relation if and only if every element of z is an ordered pair.

(d) $u = \mathrm{domain}\,(z)$ if and only if

$$\begin{aligned} &(\forall x \in u)(\exists y \in \mathcal{U}(\mathcal{U}(z)))\langle x, y\rangle \in z \\ &\quad \wedge\ (\forall x \in \mathcal{U}(\mathcal{U}(z)))(\forall y \in \mathcal{U}(\mathcal{U}(z)))(\langle x, y\rangle \in z \rightarrow x \in u) . \end{aligned}$$

(e) $v = \mathrm{range}\,(z)$ if and only if

$$\begin{aligned} &(\forall y \in v)(\exists x \in \mathcal{U}(\mathcal{U}(z)))\langle x, y\rangle \in z \\ &\quad \wedge\ (\forall x \in \mathcal{U}(\mathcal{U}(z)))(\forall y \in \mathcal{U}(\mathcal{U}(z)))(\langle x, y\rangle \in z \rightarrow y \in v) . \end{aligned}$$

(f) z is a function if and only if z is a relation and

$$(\forall x \in \mathcal{U}(\mathcal{U}(z)))(\forall y \in \mathcal{U}(\mathcal{U}(z)))(\forall y' \in \mathcal{U}(\mathcal{U}(z)))$$
$$((\langle x,y\rangle \in z \wedge \langle x,y'\rangle \in z) \to y = y') .$$

(g) $y = z(x)$ if and only if

$$(\langle x,y\rangle \in z \wedge (\exists! v \in \mathcal{U}(\mathcal{U}(z)))\, \langle x,v\rangle \in z)$$
$$\vee\ (y = \emptyset \wedge \neg(\exists! v \in \mathcal{U}(\mathcal{U}(z)))(\langle x,v\rangle \in z)) .$$

(h) z is a one-one function if and only if z is a function and

$$(\forall x \in \mathcal{U}(\mathcal{U}(z)))(\forall x' \in \mathcal{U}(\mathcal{U}(z)))(z(x) = z(x') \to x = x') . \qquad \square$$

From now on, when we state that an operation or relation is absolute for transitive models of a theory T, we mean that the operation or relation is defined in T and is absolute for transitive models of T.

Lemma 6.20. *The following operations and relations are absolute for transitive models of ZF − Power Set.*

(a) x *is an ordinal;*

(b) x *is a limit ordinal;*

(c) x *is a successor ordinal;*

(d) x *is a finite ordinal;*

(e) ω*;*

(f) $0, 1, 2 \ldots .$

Proof. (a) x is an ordinal if and only if x is transitive and $\in\restriction x$ is a linear ordering of x. The first clause is absolute by Lemma 6.12 and the second is given by a Δ_0 formula (all the quantifiers are bounded to x).

(b) x is a limit ordinal if and only if x is an ordinal and $x \neq \emptyset$ and $(\forall y \in x)\, \mathcal{S}(y) \in x$.

(c) x is a successor ordinal if and only if x is an ordinal and x is neither $\emptyset$ nor a limit ordinal.

(d) x is a natural number if and only if x is an ordinal number and neither x nor any member of x is a limit ordinal.

(e) $x = \omega$ if and only if x is a limit ordinal and no member of x is a limit ordinal.

(f) $z = 0 \leftrightarrow z = \emptyset$; $z = a + 1 \leftrightarrow (\exists x \in z)(x = a \wedge z = \mathcal{S}(x))$. $\square$

Exercise 6.2. Explain briefly which axioms of ZFC are true in the following transitive classes. (The classes are all sets, so "true in M" can be taken in either of our two senses.)

(1) V_ω ;

(2) $V_{\omega+1}$;

(3) $V_{\omega+\omega}$;

(4) V_{ω_1} ;

(5) V_κ for κ inaccessible.

A cardinal κ is *inaccessible* if κ is uncountable and regular and if, for all $\kappa' < \kappa$, $2^{\kappa'} < \kappa$.

Exercise 6.3. A formula of the language of set theory is Σ_1 if it is of the form $(\exists x_1) \ldots (\exists x_n)\, \varphi$ with φ a Δ_0 formula. A formula is Π_1 if it is of the form $(\forall x_1) \ldots (\forall x_n)\, \varphi$ with φ a Δ_0 formula. If T is a theory, a formula φ is *provably* Δ_1 *in* T if there are formulas ψ and χ such that ψ is Σ_1, χ is Π_1, and $T \models$ both $(\varphi \leftrightarrow \psi)$ and $(\varphi \leftrightarrow \chi)$.

Let φ be provably Δ_1 in T and let M and N be transitive models of T such that $M \subseteq N$. Prove that φ is absolute for (M, N).

Lemma 6.21. *Let M be a transitive model of ZF − Power Set. Then every finite subset of M belongs to M.*

Proof. There is only one subset x of M with $\operatorname{card}(x) = 0$, namely $\emptyset$, and this set belongs to M. Assume inductively that every size n subset of M belongs to M. Let $x \subseteq M$ with $\operatorname{card}(x) = n + 1$. Then there is a $y \subseteq M$ and there is a $z \in M$ such that $\operatorname{card}(y) = n$ and $x = y \cup \{z\}$. By induction $y \in M$, and so Lemma 6.12 gives that $x \in M$. $\square$

Lemma 6.22. *The following are absolute for transitive models of ZF − Power Set.*

(a) x is finite;

(b) ${}^{<\omega}x$.

Proof. (a) x is finite if and only if there is a one-one function f with $\operatorname{domain}(f) \in \omega$ and $\operatorname{range}(f) = x$. If $x \in M$, then Lemmas 6.12 and 6.20 imply that any such f is a subset of M and so, by Lemma 6.21, an element of M.

(b) We must show that $^{<\omega}x$ is defined in ZF $-$ Power Set. To do this we first use induction to prove in ZF $-$ Power Set that ^{n}x exists for every set x and every $n \in \omega$. This is true for $n = 0$, because $^{0}x = \{\emptyset\}$. It is easy to define a one-one correspondence between $^{n}x \times x$ and ^{n+1}x, so our assertion for $n+1$ follows from the assertion for n using Lemma 6.19 and Replacement. Next we use Replacement to get the existence of $\{^{n}x \mid n \in \omega\}$. Since $^{<\omega}x = \mathcal{U}\{^{n}x \mid n \in \omega\}$, we finally get the existence of $^{<\omega}x$. Absoluteness holds because $z \in {}^{<\omega}x$ if and only if z is a function and domain $(z) \in \omega$ and range $(z) \subseteq x$. $\square$

Lemma 6.23. *The following are absolute for transitive models of ZF $-$ Power Set.*

(a) r *wellorders* x*;*

(b) $\mathrm{ot}(x, r)$*, that is, the unique ordinal* α *such that* $\langle x, r\rangle$ *is isomorphic to* $\langle \alpha, \in \restriction \alpha\rangle$ *if* r *wellorders* x *and* 0 *otherwise.*

Proof. That r linearly orders x is expressible by a Δ_0 formula.

Suppose that r wellorders x. Then every non-empty subset of x has an r-least element. Let $y \in M$ be such that $(y \subseteq x)^M$ and $(y \neq \emptyset)^M$. Then $y \subseteq x$ and $y \neq \emptyset$. Let z be an r-least element of y. Then $z \in M$ and it is true in M that z is the r-least element of y.

Now suppose that "r wellorders x" is true in M. Since the proof of Theorem 1.14 goes through in ZF $-$ Power Set, it is true in M that there is an ordinal number α such that $\langle x, r\rangle$ is isomorphic to $\langle \alpha, \in \restriction \alpha\rangle$. Let f be such that in M it is true that f is an isomorphism between $\langle \alpha, \in \restriction \alpha\rangle$ and $\langle x, r\rangle$. By the absoluteness of the relevant notions, this is also true in V. Hence r wellorders x.

The argument just given proves (a), but it also shows that if r wellorders x then $\mathrm{ot}^M(x, r) = \mathrm{ot}(x, r)$. By (a) and the absoluteness of 0, we have (b). $\square$

We can extend our notion of absolute definable relations to relations defined from set parameters. For simplicity, we make this extension only for unary relations, i.e., for classes. Fix a class M. If A is the class $\{x \mid \varphi(x, a_1, \ldots, a_n)\}$, let us say that A is *defined in* M if $a_1, \ldots, a_n$ are elements of M. If A is defined in M, then

$$A^M = \{x \in M \mid \varphi^M(x, a_1, \ldots, a_n)\}.$$

We say that A is *absolute for* M if A is defined in M and $A^M = A \cap M$.

In an analogous fashion we now introduce the notion of absolute class functions. For a class function $F = \{\langle x, y\rangle \mid \varphi(\langle x, y\rangle, a_1, \ldots, a_n)\}$, let us say that F is *defined in* M *(as a function)* if $a_1, \ldots, a_n$ are elements of M and "F is a function" is true in M. If F is defined in M, then

$$F^M = \{\langle x, y\rangle \in M \mid \varphi^M(\langle x, y\rangle, a_1, \ldots, a_n)\}.$$

We say that F is *absolute for* M *(as a function)* if F is defined in M and $F^M = F \restriction M$ (so that, in particular, $\mathrm{domain}\,(F^M) = \mathrm{domain}\,(F) \cap M$).

Remarks:

(a) Being defined in M and being absolute for M depend upon the defining formula and parameters and not just on the class or function.

(b) *Definability in* M could be defined in a natural way for defined operations, although we have not done so.

(c) We have required that defined operations of n arguments be defined on any $x_1, \ldots, x_n$, but we allow absolute class functions to have domains that are not all of V.

Lemma 6.24. *Let* $F : V \to V$. *Let* $G : \mathrm{ON} \to V$ *be defined as in the proof of Theorem 1.8. Thus*

$$(\forall \alpha \in \mathrm{ON})\, G(\alpha) = F(G \restriction \alpha).$$

Let M *be a transitive model of ZF* $-$ *Power Set. Assume that* F *is absolute for* M. *Then* G *is absolute for* M.

Proof. Since the proof of Theorem 1.8 goes through in ZF $-$ Power Set and since $(F : V \to V)^M$, we have by earlier absoluteness results that G is defined in M, that $G^M : \mathrm{ON} \cap M \to M$, and that

$$(\forall \alpha \in \mathrm{ON} \cap M)\, G^M(\alpha) = F^M(G^M \restriction \alpha).$$

Using the absoluteness of F, we can prove by transfinite induction on $\alpha \in \mathrm{ON} \cap M$ that $G^M(\alpha) = G(\alpha)$. $\square$

Lemma 6.25. *The operation* trcl *is absolute for transitive models of ZF* $-$ *Power Set.*

Proof. The proof of the existence of $\mathrm{trcl}(x)$ goes through in ZF $-$ Power Set. That proof shows that $\mathrm{trcl}(x) = \mathcal{U}(\mathrm{range}\,(g_x))$ for some g_x defined by recursion from an absolute F_x (defined from x by a formula that is independent of x). Thus $\text{trcl}(x) = \mathcal{U}(\text{range}\,(G_x \restriction \omega))$ for the G_x defined by transfinite recursion from this same F_x. $\square$

For any set x, let rank (x) be the least ordinal α such that $x \in V_{\alpha+1}$. Since the V_α, $\alpha > \omega$, may not exist in models of ZF $-$ Power Set, let us adopt the following definition of rank (x) as our official definition. Given x, define by transfinite recursion a function $G_x : \mathrm{ON} \to V$ by

$$\begin{aligned} G_x(0) &= \emptyset \\ G_x(\alpha+1) &= \{y \in \text{trcl}(x) \mid y \subseteq G_x(\alpha)\}\,; \\ G_x(\lambda) &= \mathcal{U}(\text{range}\,(G_x \restriction \lambda)) \ \text{ for limit ordinals } \lambda. \end{aligned}$$

Then let rank (x) be the least α, such that $x \subseteq G_x(\alpha)$. In ZF, one can easily show that $G_x(\alpha) = V_\alpha \cap \text{trcl}(x)$, and so that the two definitions of rank (x) are equivalent in ZF.

Lemma 6.26. *The operation* rank *is absolute for transitive models of ZF $-$ Power Set.*

Lemma 6.27. *Let M be a transitive model of ZF. Then*

(a) $\mathcal{P}^M(x) = \mathcal{P}(x) \cap M$ for $x \in M$;

(b) $V_\alpha^M = V_\alpha \cap M$ for $\alpha \in \mathrm{ON} \cap M$.

Proof. (a) follows from the absoluteness of $\subseteq$. (b) follows from the absoluteness of rank. $\square$

The following lemma gives the relation between the relativization of a formula to a set and the satisfaction of that formula by the model determined by that set.

Lemma 6.28. *Let $\varphi(x_1, \ldots, x_n)$ be a formula and let b be a set. Then, for any $a_1, \ldots, a_n$ belonging to b,*

$$\varphi^b(a_1, \ldots, a_n) \leftrightarrow (b; \in) \models \varphi[a_1, \ldots, a_n]\,.$$

Proof. We can show by induction on complexity that all instances of this schema are provable. $\square$

For any set x let $\mathrm{FODO}(x)$ be the set of all $u \subseteq x$ such that, for some formula $\varphi(v_0, \ldots, v_n)$ and some sequence $\langle y_1, \ldots, y_n \rangle$ of elements of x (i.e., some $f : n \to x$),

$$u = \{y_0 \in x \mid (x; \in) \models \varphi[y_0, \ldots y_n]\}\,.$$

Lemma 6.29. *For any set x,*

(a) $\mathrm{FODO}(x) \subseteq \mathcal{P}(x)$ *;*

(b) if x is transitive, then $x \subseteq \mathrm{FODO}(x)$ *;*

(c) every finite subset of x belongs to $\mathrm{FODO}(x)$ *;*

(d) (Uses Choice) *if* $\mathrm{card}\,(x) \geq \omega$ *then* $\mathrm{card}\,(\mathrm{FODO}(x)) = \mathrm{card}\,(x)$.

Proof. (a) is obvious.

(b) Assume that x is transitive and let $b \in x$. Let $\varphi(v_0, v_1)$ be the formula $v_0 \in v_1$. Then $\{a \in x \mid (x; \in) \models \varphi[a, b]\} \in \mathrm{FODO}(x)$. But

$$\begin{aligned} \{a \in x \mid (x; \in) \models \varphi[a, b]\} &= \{a \in x \mid a \in b\} \\ &= b\,, \end{aligned}$$

where the last equality holds because x is transitive.

(c) Let $n \in \omega$ and let $u \subseteq x$ with $\mathrm{card}\,(u) = n$. Let $f : n \to u$ be one-one and onto. Then the formula $\bigvee\!\!\!\bigvee_{1 \leq i \leq n} v_0 = v_i$ and $f(0) \ldots, f(n-1)$ witness that $u \in \mathrm{FODO}(x)$.

(d) Assume that $\mathrm{card}\,(x) \geq \omega$. By (c), $\{y\} \in \mathrm{FODO}(x)$ for every $y \in x$. Thus $\mathrm{card}\,(x) \leq \mathrm{card}\,(\mathrm{FODO}(x))$. But $\mathrm{card}\,(\mathrm{FODO}(x))$ is no greater than the cardinal of $u \times v$, where u is the set of all formulas and $v = {}^{<\omega}x$. Thus $\mathrm{card}\,(\mathrm{FODO}(x)) \leq \aleph_0 \cdot \mathrm{card}\,(x) = \mathrm{card}\,(x)$. $\square$

Remark. Choice is needed for (d) only to get the existence of card (x).

By transfinite recursion, we define a function $\mathbf{L} : \mathrm{ON} \to V$. We write L_α for $\mathbf{L}(\alpha)$.

(a) $L_0 = \emptyset$;

(b) $L_{\alpha+1} = \mathrm{FODO}(L_\alpha)$;

(c) $L_\lambda = \mathcal{U}(\{L_\alpha \mid \alpha < \lambda\})$ if λ is a limit ordinal.

Let $L = \mathcal{U}(\{L_\alpha \mid \alpha \in \mathrm{ON}\})$. Members of L are said to be *constructible.*

Lemma 6.30. *For each ordinal α,*

(a) L_α is transitive;

(b) $(\forall \beta \leq \alpha)\, L_\beta \subseteq L_\alpha$.

Moreover L is transitive.

Proof. We prove (a) by transfinite induction. The case $\alpha = 0$ is trivial. The case that α is a limit ordinal follows from the fact that the union of a set of transitive sets is transitive. The case α is a successor follows from part (b) of Lemma 6.29.

The proof of (b) is just like the proof of the corresponding fact for V_α.

L is transitive because it is a union of transitive sets. $\square$

For each $x \in L$, let $\rho(x)$ (the *L-rank of x*) be the least ordinal α such that $x \in L_{\alpha+1}$.

Lemma 6.31. *(a)* $(\forall \alpha \in \mathrm{ON})(\alpha \in L \wedge \rho(\alpha) = \alpha)$ *.*

(b) $(\forall \alpha \in \mathrm{ON})\, \mathrm{ON} \cap L_\alpha = \alpha$ *.*

Proof. It is easy to see that (a) and (b) are equivalent. We prove (b) by transfinite induction.

The cases that α is 0 or a limit ordinal are trivial.

Assume that α is $\beta + 1$. Note that the proof of part (a) of Lemma 6.20 establishes that "x is an ordinal" is equivalent in ZF $-$ Power Set to a Δ_0 formula. Calling this formula $\mathrm{Ord}(x)$, we have, for $y \in L_\beta$:

$$\begin{aligned} L_\beta \models \mathrm{Ord}[y] &\leftrightarrow \mathrm{Ord}^{L_\beta}(y) \\ &\leftrightarrow \mathrm{Ord}(y) \\ &\leftrightarrow y \text{ is an ordinal} \end{aligned}$$

Thus $\mathrm{Ord}(v_0)$ witnesses that $L_\beta \cap \mathrm{ON} \in L_\alpha$ so, by induction, that $\beta \in L_\alpha$. We have then that $\mathrm{ON} \cap L_\alpha \supseteq \alpha$. But if $\gamma \geq \alpha$ then $\gamma \not\subseteq \beta$ and so $\gamma \not\subseteq L_\beta$. Thus no $\gamma \geq \alpha$ belongs to L_α. $\square$

Lemma 6.32. *For $\alpha \leq \omega$, $L_\alpha = V_\alpha$. For $n \in \omega$, V_α is finite, and so L_α is finite.*

Proof. The second assertion is easily proved by induction. The first assertion then follows by part (c) of Lemma 6.29. $\square$

Lemma 6.33. (Uses Choice) *For $\alpha \geq \omega$, $\mathrm{card}\,(L_\alpha) = \mathrm{card}\,(\alpha)$.*

Proof. By Lemma 6.31, $\mathrm{card}\,(L_\alpha) \geq \mathrm{card}\,(\alpha)$ for every α.

By transfinite induction, we show that $\mathrm{card}\,(L_\alpha) \leq \mathrm{card}\,(\alpha)$ for every $\alpha \geq \omega$.

The case $\alpha = \omega$ follows from Lemma 6.32.

For limit $\alpha > \omega$,

$$\begin{aligned}\operatorname{card}(L_\alpha) &= \operatorname{card}\Big(\bigcup_{\beta<\alpha} L_\beta\Big) \\ &\leq \operatorname{card}(\alpha \times \sup_{\beta<\alpha}\operatorname{card}(L_\beta)) \\ &\leq \operatorname{card}(\alpha \times \operatorname{card}(\alpha)) \\ &= \operatorname{card}(\alpha)\,.\end{aligned}$$

The case that α is a successor follows from part (d) of Lemma 6.29. □

Remark. Choice is not really needed for Lemma 6.33, as the proof of Theorem 6.43 will show.

Lemma 6.34. *All axioms of ZF except perhaps Comprehension hold in L.*

Proof. Extensionality holds, since L is transitive.

Foundation is trivial.

To show that Pairing holds, we use Lemma 6.5. Suppose that x and y belong to L. Let α be such that both x and y belong to L_α. Then $L_\alpha \in L_{\alpha+1} \subseteq L$, and $\{x, y\} \subseteq L_\alpha$.

For Union, we use Lemma 6.6. Let $x \in L_\alpha$. Since L_α is transitive, $x \subseteq L_\alpha$. This fact and the transitivity of L_α imply that $\mathcal{U}(x) \subseteq L_\alpha$.

For Replacement, we use Lemma 6.7. Let z and $w_1, \ldots, w_n$ belong to L and assume that

$$(\forall x \in z \cap L)(\exists! y \in L)\, \varphi^L(x, y, z, w_1, \ldots, w_n)\,.$$

By the transitivity of L and by Replacement in V, there is an α such that

$$(\forall x \in z)(\exists y \in L_\alpha)\, \varphi^L(x, y, z, w_1, \ldots, w_n)\,.$$

For Power Set, we use Lemma 6.13. Let $x \in L$. By Replacement in V, let α be such that $\mathcal{P}(x) \cap L \subseteq L_\alpha$.

For Infinity, we use Lemma 6.14 and the fact that $\omega \in L$. □

A class C of ordinals is *closed* if the union of any subset of C belongs to C. If α is an ordinal, a *closed subset* of α is a subset C of α such that the union of any subset of C bounded in α belongs to C.

Theorem 6.35 (Reflection Schema). *Let $\mathbf{M} : \mathrm{ON} \to V$. (We write M_α for $\mathbf{M}(\alpha)$.) Let $M = \bigcup_{\alpha \in \mathrm{ON}} M_\alpha$. Assume that $M_\beta \subseteq M_\alpha$ whenever $\beta \leq \alpha \in \mathrm{ON}$ and that $M_\lambda = \bigcup_{\beta<\lambda} M_\beta$ for all limit λ. Let φ be a formula. Then there is a closed, unbounded class C of ordinals such that φ is absolute for (M_α, M).*

Proof. We proceed by induction on the complexity of φ (i.e., we show inductively how the instances of the schema can be proved).

If φ is atomic, then we can let $C = \mathrm{ON}$.

If φ is $\neg\psi$ and C witnesses that the theorem holds for ψ (and $\mathbf{M}$), then C witnesses that the theorem holds for φ.

If φ is $\psi \wedge \chi$ and C' and C'' respectively witness that the theorem holds for ψ and χ, then $C = C' \cap C''$ witnesses that the theorem holds for φ.

Assume that φ is $(\exists y)\, \psi(x_1, \ldots, x_n, y)$. Define $F : {}^nV \to \mathrm{ON}$ by

$$F(\langle x_1, \ldots, x_n\rangle) = \begin{cases} \mu\alpha\, (\exists y \in M_\alpha)\, \psi^M(x_1, \ldots, x_n, y) \\ \qquad\quad \text{if } (\exists y \in M)\, \psi^M(x_1, \ldots, x_n, y)\,; \\ 0 \quad \text{otherwise.} \end{cases}$$

For ordinals α let

$$G(\alpha) = \mathcal{U}(\{F(\langle x_1, \ldots, x_n\rangle) \mid \langle x_1, \ldots x_n\rangle \in {}^nM_\alpha\})\,.$$

Let

$$C' = \{\alpha \in \mathrm{ON} \mid \alpha \text{ is a limit ordinal } \wedge\, (\forall\beta < \alpha)\, G(\beta) < \alpha\}\,.$$

The class C' is obviously closed. To see that C' is unbounded, let $\beta \in \mathrm{ON}$. Let $\beta_0 = \beta$ and, for $i \in \omega$, let $\beta_{i+1} = \max\{\beta_i, G(\beta_i)\} + 1$. If $\alpha = \bigcup_{i\in\omega} \beta_i$ then $\beta < \alpha$ and $\alpha \in C'$. Let C'' witness that the theorem holds for ψ. Let $C = C' \cap C''$. The class C is closed and unbounded. Let $\alpha \in C$ and let $x_1, \ldots, x_n$ belong to M_α. Since α is a limit ordinal, there is a $\beta < \alpha$ such that $x_1, \ldots, x_n$ belong to M_β. We have

$$\begin{aligned} \varphi^M(x_1, \ldots, x_n) &\leftrightarrow (\exists y \in M)\, \psi^M(x_1, \ldots, x_n, y) \\ &\leftrightarrow (\exists y \in M_{F(\langle x_1, \ldots, x_n\rangle)})\, \psi^M(x_1, \ldots, x_n, y) \\ &\leftrightarrow (\exists y \in M_{G(\beta)})\, \psi^M(x_1, \ldots, x_n, y) \\ &\leftrightarrow (\exists y \in M_\alpha)\, \psi^M(x_1, \ldots, x_n, y) \\ &\leftrightarrow (\exists y \in M_\alpha)\, \psi^{M_\alpha}(x_1, \ldots, x_n, y) \\ &\leftrightarrow \varphi^{M_\alpha}(x_1, \ldots, x_n)\,. \end{aligned}$$ □

Theorem 6.36. *All axioms of ZF hold in L.*

Proof. By Lemma 6.34, we need only show that comprehension holds in L. Let $\varphi(v_1, \ldots, v_{n+2})$ be a formula and let z and $w_1, \ldots, w_n$ belong to L. By Theorem 6.35, let α be an ordinal such that φ is absolute for (L_α, L) and such that z and $w_1, \ldots, w_n$ belong to L_α. We have, suppressing the w_i for brevity,

$$\begin{aligned}
\{x \in z \mid \varphi^L(x,z)\} &= \{x \in L_\alpha \mid x \in z \wedge \varphi^L(x,z)\} \\
&= \{x \in L_\alpha \mid x \in z \wedge \varphi^{L_\alpha}(x,z)\} \\
&= \{x \in L_\alpha \mid (L_\alpha; \in) \models (v_1 \in v_2 \wedge \varphi(v_1, v_2))[x,z]\} \\
&\in L_{\alpha+1} \\
&\subseteq L\,.
\end{aligned}$$

By Lemma 6.4, we have shown that Comprehension holds in L. □

Our next task is to prove that $V = L$ holds in L.

Lemma 6.37. *Every relation or function provable in ZF − Power Set to be representable in* Q *is absolute for transitive models of ZF − Power Set.*

Proof. We need to clarify the content of the lemma. When we say that, e.g., a function $f : {}^n\omega \to \omega$ is provable in ZF − Power Set to be representable in Q, we mean that f is defined (from no parameters) in ZF − Power Set and ZF − Power Set proves that some formula $\varphi(v_1, \ldots, v_{n+1})$ represents f in Q.

If φ is the representing formula, then ZF − Power Set proves that φ defines the relation or function in question in the modal $\mathfrak{N}$. Since $\mathcal{S}$, $+$, and $\cdot$ are the restrictions of functions successively definable by transfinite recursion from absolute functions, this guarantees absoluteness. □

For definiteness, let us take the symbol $\in$ to be officially the number 25.

Lemma 6.38. *The following are absolute for transitive models of ZF − Power Set:*

(a) x is a variable;

(b) $n \mapsto v_n$;

(c) $x \in$ Formula, i.e., x is a formula of the language of set theory;

(d) $\langle x, y\rangle \in$ Free, i.e., x is a formula and y is a variable occurring free in x.

Proof. (a) and (b) follow from Lemma 6.37.

For (c), note that the function $n \mapsto \mathrm{Formula}_n$ is defined by recursion from an absolute function. Formula is the union of the range of this function.

For (d), note that the function sending each n to $\{\langle x, y\rangle \in \mathrm{Free} \mid x \in \mathrm{Formula_n}\}$ is definable by recursion from an absolute function. □

Lemma 6.39. *The 3-ary relation* $\langle y, z\rangle \in \mathrm{Sat}^{(x;\in)}$ *is absolute for transitive models of ZF − Power Set.*

Proof. $n \mapsto \mathrm{Sat}_n^{(x;\in)}$ is defined by recursion from an absolute function. $\square$

Lemma 6.40. *The operation* FODO *is absolute for transitive models of ZF − Power Set.*

Proof. FODO is defined in ZF − Power Set, since Replacement guarantees the existence of $\mathrm{FODO}(x)$. Thus it is enough to show that the relation $u \in \mathrm{FODO}(x)$ is absolute. But $u \in \mathrm{FODO}(x)$ if and only if $u \subseteq x$ and

$$\begin{aligned}&(\exists\varphi)(\exists s)(\varphi \in \mathrm{Formula} \wedge s \in {}^{<\omega}x\\ &\qquad \wedge (\forall i \in \omega)(\langle\varphi, v_i\rangle \in \mathrm{Free} \to i < \ell\mathrm{h}(s)+1)\\ &\qquad \wedge (\forall y \in x)(y \in u \leftrightarrow \langle\varphi, \langle y\rangle^\frown s\rangle \in \mathrm{Sat}^{(x;\in)}))\,.\end{aligned} \qquad \square$$

Lemma 6.41. *The function* $\alpha \mapsto L_\alpha$ *is absolute for transitive models of ZF − Power Set.*

Proof. This function is defined by transfinite recursion from an absolute function. $\square$

Theorem 6.42. *The Axiom of Constructibility* $V = L$ *holds in* L*.*

Proof. We have that

$$\begin{aligned}(V = L)^L &\leftrightarrow (\forall x \in L)(\exists\alpha \in L \cap \mathrm{ON}^L)(x \in L_\alpha)^L\\ &\leftrightarrow (\forall x \in L)(\exists\alpha \in \mathrm{ON})\, x \in L_\alpha\\ &\leftrightarrow (\forall x \in L)\, x \in L\,.\end{aligned} \qquad \square$$

Theorem 6.43. *The Axiom of Choice holds in* L*.*

Proof. Fix a wellordering of Formula. By transfinite recursion, we define a function $\alpha \mapsto <_\alpha$. By induction we shall verify the following:

(i) $<_\alpha$ is a wellordering of L_α;

(ii) $(\forall x \in L_\alpha)(\forall y \in L_\alpha)(\rho(x) < \rho(y) \to x <_\alpha y)$;

(iii) $(\forall\beta < \alpha)\ <_\beta \subseteq <_\alpha$.

Set $<_0 = \emptyset$.

For α a limit ordinal, set $<_\alpha = \bigcup_{\beta<\alpha} <_\beta$. It is immediate that (iii) holds for α. The induction hypotheses that (ii) and (iii) hold for all ordinals $< \alpha$ guarantee that (ii) holds for α. Since (ii) holds for α, any failure of (i) for α would give a failure of (i) for some $\beta < \alpha$.

Assume $\alpha = \beta + 1$. For $n \in \omega$, wellorder ${}^n(L_\beta)$ lexicographically, using the ordering $<_\beta$ of L_β. (If s and t are distinct members of ${}^n(L_\beta)$, then s is less than t if, for the least m such that $s(m) \neq t(m)$, $s(m) <_\beta t(m)$.) Now order ${}^{<\omega}(L_\beta)$ by setting s less than t if $\ell\mathrm{h}(s) < \ell\mathrm{h}(t)$ or else $\ell\mathrm{h}(s) = \ell\mathrm{h}(t)$ and s is less than t in our ordering of ${}^{\ell\mathrm{h}(s)}(L_\beta)$. Finally order Formula $\times\, {}^{<\omega}(L_\beta)$ lexicographically. It is easy to check that this ordering is a wellordering. For x and y belonging to L_α, set $x <_\alpha y$ just in case one of the following holds:

(a) $x \in L_\beta \wedge y \in L_\beta \wedge x <_\beta y$;

(b) $x \in L_\beta \wedge y \notin L_\beta$;

(c) $x \notin L_\beta \wedge y \notin L_\beta$ and the least element of Formula $\times\, {}^{<\omega}(L_\beta)$ that witnesses $x \in L_\alpha$ is less than the least element that witnesses $y \in L_\alpha$.

Clearly (i), (ii), and (iii) hold for α.

Define $<_L = \bigcup_{\alpha \in \mathrm{ON}} <_\alpha$. By (i)–(iii), $<_L$ is a wellordering of L. Thus $V = L$ implies that $<_L$ wellorders V, and so $V = L$ implies Choice. Since $V = L$ holds in L, Choice holds in L. $\square$

Lemma 6.44 (Mostowski Collapse). *Let u be a set such that Extensionality holds in u. Then there is a unique transitive set v such that $(u; \in) \cong (v; \in)$. Moreover there is a unique isomorphism*

$$\pi : (u; \in) \cong (v; \in)\,.$$

Proof. For $x \in u$ we define $\pi(x)$ by recursion on rank$\,(x)$. Set

$$\pi(x) = \{\pi(y) \mid y \in x \cap u\}\,.$$

Note that this is the only possible choice of $\pi(x)$ if π is to be an isomorphism with range$\,(\pi)$ transitive.

It is clear that

$$y \in x \to \pi(y) \in \pi(x)\,.$$

To prove the converse, it is enough to show that π is one-one, and this will show that $\pi : (u; \in) \cong (\mathrm{range}\,(\pi); \in)$. By induction on the maximum of rank(x_1) and rank$(x_2)\}$, we show that $\pi(x_1) = \pi(x_2) \to x_1 = x_2$. We have

$$\begin{aligned} \pi(x_1) = \pi(x_2) &\to \{\pi(y) \mid y \in x_1 \cap u\} = \{\pi(y) \mid y \in x_2 \cap u\} \\ &\to \text{(by induction) } \{y \mid y \in x_1 \cap u\} = \{y \mid y \in x_2 \cap u\} \\ &\to \text{(by Extensionality}^u\text{) } x_1 = x_2 \,. \end{aligned}$$
□

Lemma 6.45. *Let κ be an uncountable regular cardinal. Then L_κ is a model of ZF − Power Set + $V = L$.*

Proof. Showing that L_κ is a model of ZF − Power Set will be part of a final examination problem. That $V = L$ holds in L_κ follows by Lemma 6.41. □

Lemma 6.46. *Let z be a transitive model of ZF − Power Set + $V = L$. There is an α such that $z = L_\alpha$.*

Proof. Let $\alpha = \mathrm{ON} \cap z$. Clearly α is a limit ordinal. The function $\gamma \mapsto L_\gamma$ is absolute for z. For $x \in z$ there is a $\gamma < \alpha$ such that $x \in L_\gamma$ holds in z. By absoluteness, every element of z belongs to L_α. For each $\gamma < \alpha$, $(L_\gamma)^z = L_\gamma$, and so every element of L_α belongs to z. □

Theorem 6.47. *The Generalized Continuum Hypothesis holds in L.*

Proof. Let α be an infinite ordinal number. We show that

$$\mathcal{P}(\alpha) \cap L \subseteq L_{\alpha^+} \,.$$

By Lemma 6.33, this implies that $\mathrm{card}\,(\mathcal{P}(\alpha) \cap L) \leq \alpha^+$. Hence $V = L$ implies that $2^{\mathrm{card}(\alpha)} = \alpha^+$. Since $V = L$ holds in L, the theorem will be proved.

Let $x \subseteq \alpha$ with $x \in L$. Let $\beta > \alpha$ be such that $x \in L_\beta$. By Lemma 6.45, L_{β^+} is a model of ZF − Power Set + $V = L$.

By the Löwenheim–Skolem Theorem, let y be such that

(i) $(y; \in) \prec (L_{\beta^+}; \in)$;

(ii) $\alpha \cup \{x\} \subseteq y$;

(iii) $\mathrm{card}\,(y) = \mathrm{card}\,(\alpha)$.

By Lemma 6.44, Let z and π be such that z is transitive and

$$\pi : (y; \in) \cong (z; \in) \,.$$

Since $(z; \in) \cong (y; \in) \prec (L_{\beta+}; \in)$, z is a model of ZF $-$ Power Set $+$ $V = L$. By Lemma 6.46, there is an ordinal γ such that $z = L_\gamma$.

Since $\text{card}\,(\gamma) \leq \text{card}\,(L_\gamma) = \text{card}\,(z) = \text{card}\,(y) \leq \text{card}\,(\alpha)$, we have that $\gamma < \alpha^+$.

It suffices then to prove that $x \in L_\gamma$. Since $x \in y$, we need only show that $\pi(x) = x$. First we show by induction on $\eta < \alpha$ that $\pi(\eta) = \eta$. We have

$$\begin{aligned}
\pi(\eta) &= \{\pi(\xi) \mid \xi \in \eta \cap y\} \\
&= (\text{since } \alpha \subseteq y)\ \{\pi(\xi) \mid \xi \in \eta\} \\
&= (\text{by induction})\ \{\xi \mid \xi \in \eta\} \\
&= \eta\,.
\end{aligned}$$

Finally we note that

$$\pi(x) = \{\pi(\eta) \mid \eta \in x \cap y\} = \{\pi(\eta) \mid \eta \in x\} = \{\eta \mid \eta \in x\} = x\,. \qquad \square$$

Remark. One can construct a sentence σ such that, for any transitive class M, σ holds in M if and only if $M = L$ or there is an ordinal α such that $M = L_\alpha$. Thus L_β rather than $L_{\beta+}$ could in principle have been used in the proof.

Theorem 6.48. *If ZF is consistent then so are*

(a) ZFC + $V = L$;

(b) ZFC + GCH.

Proof. Assume that ZF is consistent. Then (a) follows from Lemma 6.1 together with Lemmas 6.42 and 6.43. (b) then follows from (a) and Theorem 6.47. $\square$

The Axiom of Constructibility settles most interesting set-theoretic questions. A number of them can be answered using Jensen's combinatorial principle $\Diamond$. $\Diamond$ is the assertion that there is a sequence $\langle A_\alpha \mid \alpha < \omega_1 \rangle$ (i.e., a function $\alpha \mapsto A_\alpha$ with domain ω_1) such that each $A_\alpha \subseteq \alpha$ and such that, for any $A \subseteq \omega_1$ and any closed, unbounded subset C of ω_1,

$$(\exists \alpha \in C)\, A \cap \alpha = A_\alpha\,.$$

Theorem 6.49. $V = L \rightarrow \Diamond$.

Proof. Assume $V = L$. We define A_α by recursion. For α not a limit ordinal, set $A_\alpha = \emptyset$. Assume that α is limit ordinal and that A_β is defined for $\beta < \alpha$. Let ρ_α be the least ordinal ρ such that there are A and C belonging to L_ρ such that $A \subseteq \alpha$, C is a closed, unbounded subset of α, and

$$(\forall \alpha \in C)\, A \cap \alpha \neq A_\alpha$$

if such a ρ exists. In this case let A_α and C_α be the lexicographically least A and C (using $<_L$). If ρ_α does not exist, let $A_\alpha = \emptyset$.

Suppose that $\langle A_\alpha \mid \alpha < \omega_1 \rangle$ does not witness that $\diamondsuit$ holds. Let ρ be the least ordinal such that some counterexample sets A and C belong to L_ρ. Let A and C be the lexicographically least such pair (again using $<_L$). Note that $\rho < \omega_2$.

Let $(y; \in) \prec (L_{\omega_2}; \in)$ with y countable and with

$$\{\omega_1, \rho, A, C, \langle A_\beta \mid \alpha < \omega_1 \rangle\} \subseteq y\,.$$

Let z and π be such that z is transitive and $\pi : (y; \in) \cong (z; \in)$. Let $\delta < \omega_1$ be such that $z = L_\delta$.

Let $\alpha = \pi(\omega_1)$. By final examination problem 4(a), we have that $\alpha \subseteq y$. It follows that

(i) $A \cap \alpha = \pi(A)$;

(ii) $C \cap \alpha = \pi(C)$;

(iii) $\langle A_\beta \mid \beta < \alpha \rangle = \pi(\langle A_\beta \mid \beta < \omega_1 \rangle)$.

Using (i)–(iii), the definitions of ρ, A, and C, and the fact that π^{-1} is an elementary embedding of $(L_\delta; \in)$ into $(L_{\omega_2}; \in)$, we get that $\pi(\rho)$, $A \cap \alpha$, and $C \cap \alpha$ satisfy in L_δ the definitions of ρ_α, A_α, and C_α respectively. Thus

(a) $\pi(\rho) = \rho_\alpha$;

(b) $A \cap \alpha = A_\alpha$;

(c) $C \cap \alpha = C_\alpha$.

Since $C \cap \alpha = \pi(C)$, $C \cap \alpha$ is an unbounded subset of α. Since C is closed, it follows that $\alpha \in C$. This fact and (b) contradict the definitions of A and C. $\square$

One of the earliest applications of $\diamondsuit$ was to show that *Souslin's Hypothesis* fails in L.

To state Souslin's Hypothesis, we need some definitions. Let R be a linear ordering of a set X. If every R-bounded subset of X has a least upper bound, then $(X;R)$ is said to be *complete*. If every set of disjoint open (in the obvious sense) R-intervals is countable, then $(X;R)$ is *ccc*: *satisfies the countable chain condition.* Give X the order topology: the basic open sets are the open intervals. If X has a countable dense subset then $(X;R)$ is *separable.*

The set $\mathbb{R}$ of reals, with its usual ordering, is—up to isomorphism—the unique separable, complete, dense linear ordering without endpoints. Souslin's hypothesis says this characterization continues to hold when "separable" is replaced by "ccc." Clearly the failure of Souslin's Hypothesis is equivalent to the existence of a *Souslin line*, a ccc, complete, dense linear ordering that is not separable.

The existence of a Souslin line is can be shown equivalent to the existence of a *Souslin tree*: a $(T;\lhd)$ such that

(1) $\lhd$ is a partial ordering of T;

(2) For all $x \in T$, $\{y \in T \mid x \lhd y\}$ is wellordered by $\lhd$;

(3) $\mathrm{card}(T) = \aleph_1$;

(4) $(T;\lhd)$ has no uncountable branches and no uncountable antichains.

Here a *branch* is a maximal subset of T linearly ordered by $\lhd$, and an *antichain* is a set of pairwise $\lhd$-incomparable elements of T.

Conditions (1) and (2) define the (set-theoretic) concept of a *tree.* Let us call a tree $(T;\lhd)$ *ultranormal* if

(i) $T \subseteq \omega_1$;

(ii) for β and $\gamma \in T$, $\beta \lhd \gamma \to \beta < \gamma$;

(iii) T has a $\lhd$-least element;

(iv) For each $\alpha < \omega_1$, the set of all $\beta \in T$ such that $\mathrm{level}(\beta) = \alpha$ is countable, where $\mathrm{level}(\beta)$ is the $\lhd$ order type of $\{\gamma \in T \mid \gamma \lhd \beta\}$;

(v) if $\beta \in T$ then β has infinitely many immediate successors with respect to $\lhd$;

(vi) for each $\beta \in T$ and each α such that $\mathrm{level}(\beta) < \alpha < \omega_1$, there is a $\gamma \in T$ such that $\mathrm{level}(\gamma) = \alpha$ and $\beta \lhd \gamma$;

(vii) if β and γ are elements of T with the same limit level and the same $\lhd$-predecessors, then $\beta = \gamma$.

Lemma 6.50. *If there is an ultranormal Souslin tree, then there is a Souslin line.*

Proof. We first observe that it is enough to construct a ccc, dense, linear ordering $(X;R)$ that is not separable. If we have such an $(X;R)$, then we can let X' be the set of all *Dedekind cuts* in $(X;R)$, i.e., the set of all bounded initial segments of $(X;R)$ without R-greatest elements, and we can let $x' \, R' \, y' \leftrightarrow x' \subseteq y'$. Clearly $(X';R')$ a linear ordering. The function $x \mapsto \{y \in X \mid y \, R \, x\}$ embeds $(X;R)$ into $(X';R')$ and has dense range. Therefore $(X';R')$ is dense, ccc, and not separable. If A is an R'-bounded subset of X', then $\bigcup R'$ is the least upper bound of A; hence $(X';R')$ is complete.

Let $(T;\lhd)$ be an ultranormal Souslin tree. Let

$$X = \{b \mid b \text{ is a branch of } T\}\,.$$

To define an ordering R on X, let us first fix, for each $\beta \in T$, an ordering $<_\beta$ of the the immediate successors of β with respect to $\lhd$. By (iv) and (v), we can—and do—make $<_\beta$ isomorphic to the standard ordering of the rationals. Let b and b' be distinct branches of $(T;\lhd)$. By (vii), there is a $\lhd$-greatest β that belongs to both b and b'. Let γ and γ' be the immediate $\lhd$-successors of β that belong to b and b' respectively. Define

$$b \, R \, b' \;\leftrightarrow\; \gamma <_\beta \gamma'.$$

It is easy to see that R is a linear ordering of X. Suppose that I is an open interval of $(X;R)$. let $I = (b, b')$. Define β, γ, and γ' as in the preceding paragraph. Let δ_I be such that $\gamma <_\beta \delta_I <_\beta \gamma'$. Observe that every branch containing δ_I belongs to the interval I. Observe also that if I_1 and I_2 are disjoint intervals, then δ_{I_1} and δ_{I_2} are $\lhd$-incomparable. The first fact implies that the $(X;R)$ is a dense ordering, and the second fact implies that $(X;R)$ has the ccc. For non-separability, let B be any countable subset of X. Since every member of B is countable, $\bigcup_{b\in B} b$ is countable. Let $\alpha \in T$ be $>$ every member of this countable set. Then the set of branches containing α is a neighborhood witnessing that B is not dense. $\square$

Theorem 6.51. *If $\diamondsuit$ holds, then there is an ultranormal Souslin tree.*

Proof. Let $\langle A_\alpha \mid \alpha < \omega_1\rangle$ witness that $\diamondsuit$ holds.

We shall define an ultranormal tree $(T;\lhd)$ by transfinite recursion. More precisely, we shall define for each $\alpha < \omega_1$ a tree $(T_\alpha;\lhd_\alpha)$, and we shall arrange that

(a) for $\alpha' < \alpha < \omega_1$, $T_{\alpha'}$ is the set of all elements of T_α of $\lhd_\alpha$-level $\leq \alpha'$, and $\lhd_{\alpha'}$ is the restriction of $\lhd_\alpha$ to T'_α;

(b) for $\alpha < \omega_1$, (i)-(vii) hold with $(T_\alpha; \lhd_\alpha)$ replacing $(T; \lhd)$ and with the $\alpha + 1$ replacing ω_1..

We shall then let $T = \bigcup_{\alpha<\omega_1} T_\alpha$ and $\lhd = \bigcup_{\alpha<\omega_1} \lhd_\alpha$. The only task that will remain to us is the verification that $(T; \lhd)$ satisfies condition (4) in the definition of a Souslin tree.

Let $\alpha < \omega_1$ and assume that $(T_{\alpha'}; \lhd_{\alpha'})$ is defined for $\alpha' < \alpha$ in such a way that (a) and (b) are not violated.

If $\alpha = 0$ let $T_0 = \{0\}$ and stipulate that 0 does not bear $\lhd_0$ to itself.

If $\alpha = \alpha' + 1$ for some α', then assign to the ordinals $\beta \in T_{\alpha'}$ of level α' disjoint countable infinite sets $B_\beta \subseteq \omega_1$. Do this so that $\beta < \gamma \notin T_{\alpha'}$ for each $\gamma \in B_\beta$. Let

$$T_\alpha = T_{\alpha'} \cup \bigcup \{B_\beta \mid \beta \in T_{\alpha'} \wedge \text{level}(\beta) = \alpha'\}\,.$$

Let

$$\lhd_\alpha = \lhd'_\alpha \cup \{\langle \beta, \gamma\rangle \mid \beta \in T_{\alpha'} \wedge \text{level}(\beta) = \alpha' \wedge \gamma \in B_\beta\}\,.$$

Assume that α is a limit ordinal. Let $\langle \alpha_i \mid i \in \omega\rangle$ be a strictly increasing sequence of ordinals with supremum α. Let

$$\begin{array}{lcl} T^*_\alpha & = & \bigcup_{\alpha'<\alpha} T_{\alpha'} \quad (= \quad \bigcup_{i\in\omega} T_{\alpha_i}); \\ \lhd^*_\alpha & = & \bigcup_{\alpha'<\alpha} \lhd_{\alpha'} \quad (= \quad \bigcup_{i\in\omega} \lhd_{\alpha_i}). \end{array}$$

For $\beta \in T^*_\alpha$, define $\langle \beta_i \mid i \in \omega\rangle$ by recursion as follows. If A_α is not a maximal antichain in the tree $(T^*_\alpha; \lhd^*_\alpha)$ or if there is a $\xi \in A_\alpha$ such that $\xi \lhd^*_\alpha \beta$, then set $\beta_0 = \beta$. Otherwise there is a $\xi \in A_\alpha$ such that $\beta \lhd^*_\alpha \xi$. Let β_0 be some such ξ. If $\text{level}(\beta_i) \geq \alpha_i$, then let $\beta_{i+1} = \beta_i$. If $\text{level}(\beta_i) < \alpha_i$, let $\beta_{i+1} \in T_{\alpha_i}$ be such that $\beta_i \lhd_{\alpha_i} \beta_{i+1}$ and $\text{level}(\beta_{i+1}) = \alpha_i$. (Such a β_{i+1} exists by condition (vi) on $(T_{\alpha_i}; \lhd_{\alpha_i})$.) Let b_β be the unique branch containing all the β_i. Let $\mathcal{B}_\alpha$ be the set of all the b_β for $\beta \in T^*_\alpha$. For each $b \in \mathcal{B}_\alpha$, let γ_b be a countable ordinal γ such that $\gamma \notin T^*_\alpha$ and $\gamma >$ every member of b. Make sure that the function $b \mapsto \gamma_b$ is one-one. Let

$$T_\alpha = T^*_\alpha \cup \{\gamma_b \mid b \in \mathcal{B}_\alpha\}\,.$$

Let

$$\lhd_\alpha = \lhd^*_\alpha \cup \{\langle \delta, \gamma_b\rangle \mid (b \in \mathcal{B}_\alpha \wedge \delta \in b)\}\,.$$

To verify that $(T; \lhd)$ satisfies condition (4), we first show that if $(T; \lhd)$ has an uncountable branch then it has an uncountable antichain. Let b be an uncountable branch. By condition (v), each $\beta \in b$ has an immediate $\lhd$-successor that does not belong to b. Let

$$A = \{\gamma \mid \gamma \notin b \ \wedge \ (\exists \beta \in b)\, \gamma \text{ is an immediate } \lhd\text{-successor of } \beta\}\,.$$

The uncountable set A is clearly an antichain of $(T; \lhd)$.

Since every antichain can be extended to a maximal antichain, it suffices to prove that $(T; \lhd)$ has no uncountable maximal antichains.

Let A be a maximal antichain of $(T; \lhd)$. For limit $\alpha < \omega_1$, let $(T^*_\alpha; \lhd^*_\alpha)$ be defined as above. Note that T^*_α is the set of $\beta \in T$ such that, with respect to $\lhd$, level$(\beta) < \alpha$. Note also that $\lhd^*_\alpha$ is just the restriction of $\lhd$ to T^*_α.

Let C be the set of all limit $\alpha < \omega_1$ such that

(a) $T^*_\alpha = T \cap \alpha$;

(b) $A \cap \alpha$ is a maximal antichain of $(T^*_\alpha; \lhd^*_\alpha)$.

We shall prove that C is closed and unbounded in ω_1.

By the definition of T^*_α, it is clear that $\{\alpha \mid T^*_\alpha = T \cap \alpha\}$ is closed in ω_1. To show that C is closed, it is therefore enough to show that the set of all α that satisfy (b) is closed in ω_1. Suppose that $\langle \alpha_i \mid i \in \omega\rangle$ is a strictly increasing sequence of countable ordinals such that for each i, $A \cap \alpha_i$ a maximal antichain of $(T^*_{\alpha_i}; \lhd^*_{\alpha_i})$. Let $\alpha = \bigcup_{i\in\omega} \alpha_i$. Let $\beta \in T^*_\alpha$. For any sufficiently large $i \in \omega$, $\beta \in T^*_{\alpha_i}$. Thus β is comparable with some $\gamma \in A \cap \alpha_i \subseteq A \cap \alpha$. This shows that $A \cap \alpha$ is a maximal antichain in $(T^*_\alpha; \lhd^*_\alpha)$.

For $\alpha < \omega_1$, let

$$\begin{aligned} f(\alpha) &= \mu\delta\, (\forall \beta \in T^*_\alpha)\, \beta < \delta; \\ g(\alpha) &= \mu\delta\, (\forall \beta \in T^*_\alpha)(\exists \gamma \in A \cap \delta)\, \gamma \text{ is } \lhd\text{-comparable with } \beta. \end{aligned}$$

That $f(\alpha)$ and $g(\alpha)$ are defined for every α follows from the fact that T^*_α is countable (by (iv)) and the fact that A is an maximal antichain of $(T; \lhd)$. By an argument like one in the proof of Theorem 6.35, the set C' of all countable ordinals closed under f and g is an unbounded subset of ω_1. By (ii), $T \cap \alpha \subseteq T^*_\alpha$ for every $\alpha < \omega_1$. Therefore every $\alpha \in C'$ satisfies (a) and (b).

Since $\langle A_\alpha \mid \alpha < \omega_1\rangle$ witnesses the truth of $\Diamond$, let $\alpha \in C$ be such that $A \cap \alpha = A_\alpha$. By (b), A_α is a maximal antichain of T^*_α. By the definition of $\mathcal{B}_\alpha$, every $b \in \mathcal{B}_\alpha$ contains a member of A_α. For $b \in \mathcal{B}_\alpha$, every member of b is $\lhd_\alpha \gamma_b$ and so is $\lhd \gamma_b$. Hence for each $b \in \mathcal{B}_\alpha$ there is a $\xi \in A_\alpha$ such that $\xi \lhd \gamma_b$. If $\beta \in T \setminus T_\alpha$, then there is a b such that $\gamma_b \lhd \beta$. Putting all these facts together, we get that every element of T is $\lhd$-comparable with some element of A_α. In other words, A_α—i.e., $A \cap \alpha$—is a maximal antichain of T. But this means that $A = A \cap \alpha$. Hence A is countable. $\square$