230B: Homework 2

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1. Dummit and Foote exercises

Exercise 1 (Dummit and Foote: 8.1.7)
Find a generator for the ideal $(47-13i, 53+56i)$ in $\mathbf{Z}[i]$, that is, find the greatest common divisor of $47-13i$ and $53+56i$.

Exercise 2 (Dummit and Foote: 8.1.8; see hints in book)
Let $F = \mathbf{Q}(\sqrt{D})$ be a quadratic field with associated quadratic integer ring $\mathcal{O}$ and field norm N (see Dummit and Foote, section 7.1).
(a) Suppose D is $-1, -2, -3, -7$ or -11. Prove that $\mathcal{O}$ is a Euclidean domain with respect to N.
(b) Suppose that $D = -43$. Prove that $\mathcal{O}$ is not a Euclidean norm with respect to any norm.

Exercise 3 (Dummit and Foote: 8.2.6)
Let R be an integral domain and suppose that every prime ideal in R is principal. This exercise proves that every ideal of R is principal, i.e., R is a P.I.D.
(a) Assume that the set of ideals of R that are not prinicipal is nonempty and prove that this set has a maximal element under inclusion (hint: use Zorn).
(b) Let I be an ideal which is maximal with respect to being nonprincipal, and let $a, b \in R$ with $ab \in I$ but $a \notin I$ and $b \notin I$. Let $I_a = (I, a)$ be the ideal generated by I and a, let $I_b = (I, b)$ be the ideal generated by I and b, and define $J = \{r \in R : rI_a \subseteq I\}$. Prove that $I_a = (\alpha)$, and $J = (\beta)$ are principal ideals in R with $I \subsetneq I_b \subseteq J$ and $I_a J = (\alpha\beta) \subseteq I$.
(c) If $x \in I$ show that $x = s\alpha$ for some $s \in J$. Deduce that $I = I_a J$ is principal, a contradiction, and conclude that R is a P.I.D.

Exercise 4 (Dummit and Foote: 8.2.8)
Prove that if R is a P.I.D. and D is a multiplicatively closed subset of R (not containing 0), then $D^{-1}R$ is a P.I.D.

Exercise 5 (Dummit and Foote: 8.3.3)
Determine all the representations of the integer $2130797 = 17^2 \cdot 73 \cdot 101$ as the sum of two squares.

Exercise 6 (Dummit and Foote: 8.3.11; see hints in book)
Prove that R is a P.I.D if and only if R is a U.F.D. that is also a Bezout Domain (see exercise 8.2.7).

2. Qualifying exam exercises

The problems below are (literally) taken from old qualifying exams from UCI.

Exercise 7
A commutative ring R with identity $1 \neq 0$ is called boolean if $x^2 = x$ for every $x \in R$.
(a) Find all boolean integral domains.
(b) Prove that every prime ideal in a boolean ring is maximal.

Exercise 8
For which primes p can one find a nonzero homomorphism $\mathbf{Z}[i] \to \mathbf{Z}/p\mathbf{Z}$?

Exercise 9
Let R be the ring $\mathbf{Z}[\sqrt{-5}]$.
(a) Show that R is not a UFD.
(b) Factor the principal ideal (6) into a product of prime ideals in the ring R.

Exercise 10
Determine whether each of the following statements is true or false, and justify your answer with a proof or counterexample (justify your counterexample). (a) The groups $\mathbf{Z}/20\mathbf{Z} \times \mathbf{Z}/6\mathbf{Z}$ and $\mathbf{Z}/12\mathbf{Z} \times \mathbf{Z}/10\mathbf{Z}$ are isomorphic.
(b) The group of units in $\mathbf{Z}/12\mathbf{Z}$ is isomorphic to $\mathbf{Z}/4\mathbf{Z}$.
(c) Every UFD is a PID.
(d) For every commutative ring R, every subring of R is an ideal of R.
(e) For every commutative ring R, every ideal of R is a subring of R.
(f) For every commutative ring R with unity, every prime ideal of R is a maximal ideal of R.