230B: Homework 3

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1. Dummit and Foote exercises

Exercise 1 (Dummit and Foote: 9.1.7)
Let R be a commutative ring with 1. Prove that a polynomial ring in more than one variable over R is not a Principal Ideal Domain.

Exercise 2 (Dummit and Foote: 9.2.2)
Let F be a finite field of order q and let $f(x)$ be a polynomial in $F[X]$ of degree $n \geq 1$. Prove that $R[X]/(f(x))$ has q^n elements.

Exercise 3 (Dummit and Foote: 9.3.2)
Prove that if $f(x)$ and $g(x)$ are polynomials with rational coefficients whose product $f(x)g(x)$ has integer coefficients, then the product of any coefficient of $g(x)$ with any coefficient of $f(x)$ is an integer.

Exercise 4 (Dummit and Foote: 9.3.4)
Let $R = \mathbf{Z} + x\mathbf{Q}[X] \subset \mathbf{Q}[X]$ be the set of polynomials in x with rational coefficients whose constant term is an integer.
(a) Prove that R is an integral domain and its units are ± 1.
(b) Show that the irreducibles in R are $\pm p$ where p is a prime in $\mathbf{Z}$ and the polynomials $f(x)$ that are irreducible in $\mathbf{Q}[X]$ and have constant term ± 1. Prove that these irreducibles afre prime in R.
(c) Show that x cannot be written as the product of irreducibles in R (in particular, x is not irreducible) and conclude that R is not a U.F.D.
(d) Show that x is not a prime in R and describe the quotient ring $R/(x)$.

Exercise 5 (Dummit and Foote: 9.4.1)
Determine whether the following polynomials are irreducible in the rings indicated. For those that are irreducible, determine their factorization into irreducibles. The notation $\mathbf{F}_p$ denoted the finite field $\mathbf{Z}/p\mathbf{Z}$, p a prime.
(a) $x^2 + x + 1$ in $\mathbf{F}_2[x]$.
(b) $x^3 + x + 1$ in $\mathbf{F}_3[x]$.
(c) $x^4 + 1$ in $\mathbf{F}_5[x]$.
(d) $x^4 + 10x^2 + 1$ in $\mathbf{Z}[x]$.

Exercise 6 (Dummit and Foote: 9.4.2)
Prove that the following polynomials are irreducible in $\mathbf{Z}[x]$:
(a) $x^4 - 4x^3 + 6$
(b) $x^6 + 30x^5 - 15x^3 + 6x - 120$
(c) $x^4 + 4x^3 + 6x^2 + 2x + 1$ (hint: substitute $x - 1$ for x.)
(d) $\frac{(x+2)^p - 2^p}{x}$, where p is an odd prime.

Exercise 7 (Dummit and Foote: 9.5.4)
Prove that $x^3 + 12x^2 + 18x + 6$ is irreducible over $\mathbf{Z}[i]$. (hint: use Proposition 8.18 and Eisenstein's Criterion.).

2. Qualifying exam exercises

The problems below are (literally) taken from old qualifying exams from UCI.

Exercise 8
Let F be a field. Prove that $\langle F, +\rangle$ and $\langle F^{\times}, \cdot\rangle$ are not isomorphic as groups.

Exercise 9
Prove or disprove: The quotient ring $\mathbf{F}_2[x]/(x^4+x^3+x^2+x+1)$ is a field.

Exercise 10
Let F be a field. Prove that the ring $F[x^2, x^3]$ is not a unique factorization domain.

Exercise 11
Prove that a finite integral domain is a field.