230B: Homework 4

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1. Dummit and Foote exercises

Rings R below will always have a 1.

Exercise 1 (Dummit and Foote: 10.1.8)
An element m of the R-module M is called a torsion element if $rm = 0$ for some nonzero element $r \in R$. The set of torsion elements is denoted by

$$\mathrm{Tor}(M) = \{m \in M \mid rm = 0 \text{ for some nonzero } r \in R\}.$$

(a) Prove that if R is an integral domain then $\mathrm{Tor}(M)$ is a submodule of M (called the torsion submodule of M).
(b) Give an example of a ring R and an R-module M such that $\mathrm{Tor}(M)$ is not a submodule. (hint: consider the torsion elements in the R-module R).
(c) If R has zero divisors show that every nonzero R-module has nonzero torsion elements.

Exercise 2 (Dummit and Foote: 10.2.13)
Let I be a nilpotent ideal in a commutative ring (see Exercise 7.3.37), let M and N be R-modules and let $\varphi : M \to N$ be an R-module homomorphism. Show that if the induced map $\overline{\varphi} : M/IM \to N/IN$ is surjective, then φ is surjective.

Exercise 3 (Dummit and Foote: 10.3.10)
Assume R is commutative. Show that an R-module M is irreducible (see Exercise 10.3.9: M has precisely 2 submodules) if and only if M is isomorphic as R-module to R/I where I is a maximal ideal of R.

Exercise 4 (Dummit and Foote: 10.3.11)
Show that if M_1 and M_2 are irreducible R-modules, then any nonzero R-module homomorphism from M_1 to M_2 is an isomorphism. Deduce that if M is irreducible then $\mathrm{End}_R(M)$ is a division ring.

Exercise 5 (Dummit and Foote: 10.3.24) (direct product of free is not always free)
For each positive integer i let M_i be the free $\mathbf{Z}$-module $\mathbf{Z}$, and let M be the direct product $\prod_{i \in \mathbf{Z}_{\geq 1}} M_i$ and consider the submodule $N = \bigoplus_{i \in \mathbf{Z}_{\geq 1}} M_i$ (direct sum). Assume that M is a free $\mathbf{Z}$-module with basis $\mathfrak{B}$.
(a) Show that N is countable.
(b) Show that there is some countable subset $\mathfrak{B}_1$ of $\mathfrak{B}$ usch that N is contained in the submodule N_1 generated by $\mathfrak{B}_1$. Show also that N_1 is countable.
(c) Let $\overline{M} = M/N_1$. Show that $\overline{M}$ is a free $\mathbf{Z}$-module. Deduce that if $\overline{x}$ is any nonzero element of $\overline{M}$ then there are only finitely many distinct positive integers k such that $\overline{x} = k\overline{m}$ for some $m \in M$ (depending on k).
(d) Let $\mathfrak{S} = \{(b_1, b_2, b_3, \ldots) : b_i = \pm i! \text{ for all } i\}$. Prove that $\mathfrak{S}$ is uncountable. Deduce that there is some $s \in \mathfrak{S}$ with $s \notin N_1$.
(e) Show that the assumption M is free leads to a contradiction: By (d) we may choose $s \in \mathfrak{S}$ with $s \notin N_1$. Show that for each positive integer k there is some $m \in M$ with $\overline{s} = k\overline{m}$, contrary to (c).

Exercise 6 (Dummit and Foote: 10.5.1ace)
Suppose that

$$\begin{tikzcd}
A \arrow[r, "\psi"] \arrow[d, "\alpha"] & B \arrow[r, "\varphi"] \arrow[d, "\beta"] & C \arrow[d, "\gamma"] \\
A' \arrow[r, "\psi'"] & B' \arrow[r, "\varphi'"] & C'
\end{tikzcd}$$

is a commutative diagram of groups and that the rows are exact. Prove that:
(a) if φ and α are surjective, and β is injective, then γ is injective.
(c) if φ, α and γ are surjective, then β is surjective.
(e) if β is surjective, γ and ψ' are injective, then α is surjective.

2. Other exercises

Let $f : M \to N$ be an R-module homomorphism. We let $\mathrm{coker}(f) = N/\mathrm{im}(f)$ be the cokernel.

Exercise 7 (Snake lemma; counts as 2 exercises)
Let R be a commutative ring with 1. Assume that we have the following commutative diagram of R-modules with exact rows:

$$\begin{tikzcd}
& A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C'. &
\end{tikzcd}$$

Show that there is an exact sequence

$$\ker(a) \to \ker(b) \to \ker(c) \to \mathrm{coker}(a) \to \mathrm{coker}(b) \to \mathrm{coker}(c).$$