230B: Homework 5

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1. Dummit and Foote exercises

Exercise 1 (Dummit and Foote: 10.4.4)
Show that $\mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{Q}$ and $\mathbf{Q} \otimes_{\mathbf{Q}} \mathbf{Q}$ are isomorphic as left $\mathbf{Q}$-modules (hint: show they are both 1-dimensional vector spaces over $\mathbf{Q}$).

Exercise 2 (Dummit and Foote: 10.4.6)
If R is any integral domain with quotient field Q, prove that $(Q/R) \otimes_R (Q/R) = 0$.

Exercise 3 (Dummit and Foote: 10.4.8)
Suppose that R is an integral domain with quotient field Q and let N be any R-module. Let $U = R^*$ be the set of nonzero elements of R and define $U^{-1}N$ to be the set of equivalence classes of ordered pairs of elements (u, n) with $u \in U$ and $n \in N$ under the equivalence relation $(u, n) \sim (u', n')$ if and only if $xu'n = xun'$ in N for some $x \in U$ (Note: correction).
(a) Prove that $U^{-1}N$ is an abelian group under the addition defined by $\overline{(u_1, n_1)} + \overline{(u_2, n_2)} = \overline{(u_1u_2, u_2n_1 + u_1n_2)}$. Prove that $r\overline{(u, n)} = \overline{(u, rn)}$ defines an action of R of $U^{-1}N$ making it into an R-module.
(b) Show that the map $Q \times N$ to $U^{-1}N$ defined by sending $(a/b, n)$ to $\overline{(b, an)}$ for $a \in R$, $b \in U$, $n \in N$, is an R-balanced map, so induces a homomorphism f from $Q \otimes_R N$ to $U^{-1}N$. Show that the map g from $U^{-1}N$ to $Q \otimes_R N$ defined by $g(\overline{(u, n)}) = (1/u) \otimes n$ is well defined and is an inverse homomorphism to f. Conclude that $Q \otimes_R N \cong U^{-1}N$ as R-modules.
(c) Conclude from (b) that $(1/d) \otimes n$ is 0 in $Q \otimes_R N$ if and only if $rn = 0$ for some nonzero $r \in R$.
(d) If A is an abelian group, show that $\mathbf{Q} \otimes_{\mathbf{Z}} A = 0$ if and only if A is a torsion abelian group (i.e., every element of A has finite order).

Exercise 4 (Dummit and Foote: 10.4.11)
Let $\{e_1, e_2\}$ be a basis of $V = \mathbf{R}^2$. Show that the element $e_1 \otimes e_2 + e_2 \otimes e_1$ in $V \otimes_R V$ cannot be written as a simple tensor $v \otimes w$ for any $v, w \in R^2$.

Exercise 5 (Dummit and Foote: 10.4.15)
Show that tensor products do not commute with direct products in general. (hint: consider the extension of scalars from $\mathbf{Z}$ to $\mathbf{Q}$ of the direct product of the modules $M_i = \mathbf{Z}/2^i\mathbf{Z}$, $i = 1, 2, \ldots$).

2. Qualifying exam exercises

Exercise 6
Suppose that R is a commutative ring with identity and M_1, M_2 are distinct maximal ideals of R. Show that $R/M_1 \otimes_R R/M_2 = 0$.

Exercise 7
For relatively prime positive integers m and n, show that

$$\mathbf{Z}/m\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Z}/n\mathbf{Z} = 0.$$

Exercise 8
True/False. For each of the following answer True or False and give a brief explanation.
(a) If K_1, K_2 are fields and $\varphi : K_1 \to K_2$ is a ring homomorphism such that

$\varphi(1) = 1$, then φ is injective.
(b) The unit group of $\mathbf{C}$ is isomorphic to the additive group of $\mathbf{C}$.
(c) Let n be a positive integer. Then $\mathbf{Z}/n\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Q} = 0$.

Exercise 9
For each of the following, either give an example or state that none exists. In either case, give a brief explanation.
(a) A non-zero zero divisor in $\mathbf{C} \otimes_R \mathbf{C}$.
(b) An injective group homomorphism $(\mathbf{Z}/8\mathbf{Z})^* \to \mathbf{Z}/24\mathbf{Z}$.