230B: Homework 6

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1. Dummit and Foote exercises

Exercise 1 (Dummit and Foote: 12.1.6)
Show that if R is an integral domain and if M is any nonprincipal ideal of R then M is torsion free of rank 1 but it is not a free R-module.

Exercise 2 (Dummit and Foote: 12.2.9)
Find the rational canonical form of

$$\begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

Exercise 3 (Dummit and Foote: 12.2.17)
Determine representatives for the conjugacy class for $\mathrm{GL}_3(\mathbf{F}_2)$.

Exercise 4 (Dummit and Foote: 12.3.16)
Determine the Jordan canonical form for the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and determine a matrix P which conjugates this matrix into its Jordan canonical form.

Exercise 5 (Dummit and Foote: 12.3.24)
Prove there are no 3×3 matrices A over $\mathbf{Q}$ with $A^8 = I$ but $A^4 \neq I$.

2. Qualifying exam exercises

Exercise 6
Classify, up to conjugation, all 4×4 real matrices A which satisfy $A^3 = I$ where I is the identity matrix.

Exercise 7
(a) Determine all real matrices A with characteristic polynomial $X^3(X^2+1)$.
(b) (ADDED PART) Determine all complex matrices A with characteristic polynomial $X^3(X^2+1)$.

Exercise 8
Classify, up to conjugation, all 4×4 real matrices with minimal polynomial $(X^2+4)(X-1)$.

Exercise 9
Suppose that A is a nilpotent matrix. Show that $\det(A+I) = 1$.

Exercise 10 If V is a vector space and $V = A \oplus B = C \oplus D$ with $A \cong C$, does it follow that $B \cong D$? Justify your answer.

Exercise 11 Suppose that V is a vector space and let $\mathrm{GL}(V)$ be the group of all invertible linear transformations from V to itself. Suppose G is a subgroup of

$\mathrm{GL}(V)$ and define R to be the set of all linear transformations $T : V \to V$ such that $T(g(v)) = g(T(v))$ for every $g \in G$ and $v \in V$.
(a) Show that R is a ring.
(b) Suppose further that if W is any subspace of V such that $g(W) \subset W$ for every $g \in G$, then either $W = 0$ or $W = V$. Prove that if $T \in R$ and T is not the zero transformation, then T is invertible and $T^{-1} \in R$. (Hint: if $T \in R$, what can you say about the kernel and image of T?).