230B: Homework 7

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1. Dummit and Foote exercises

Exercise 1 (Dummit and Foote: 13.1.2)
Show that $x^3 - 2x + -2$ is irreducible over $\mathbf{Q}$ and let θ be a root. Compute $(1+\theta)(1+\theta+\theta^2)$ and $\frac{1+\theta}{1+\theta+\theta^2}$ in $\mathbf{Q}(\theta)$.

Exercise 2 (Dummit and Foote: 13.2.13)
Prove that if $[F(\alpha) : F]$ is odd then $F(\alpha) = F(\alpha^2)$.

Exercise 3 (Dummit and Foote: 13.2.16 and some extra)
(a) Let K/F be an algebraic extension and let R be a ring contained in K and containing F. Show that R is a subfield of K containing F.
(b) Find a counterexample to (a) when K/F is not algebraic.

Exercise 4 (Dummit and Foote: 13.2.22)
Let K_1 and K_2 be two finite extensions of a field F contained in a field K. Prove that the F-algebra $K_1 \otimes_F K_2$ is a field if and only if $[K_1K_2 : F] = [K_1 : F][K_2 : F]$.

Exercise 5 (Dummit and Foote: 13.4.1)
Determine the splitting field and its degree over $\mathbf{Q}$ for $x^4 - 2$.

Exercise 6 (Dummit and Foote: 13.4.2)
Determine the splitting field and its degree over $\mathbf{Q}$ for $x^4 + 2$.

Exercise 7 (Dummit and Foote: 13.5.2)
Find all irreducible polynomials of degrees $1, 2$ and 4 over $\mathbf{F}_2$ and prove that their product is $x^{16} - x$.

Exercise 8 (Dummit and Foote: 13.5.5)
For any prime p and any nonzero $a \in \mathbf{F}_p$ prove that $x^p - x + a$ is irreducible and separable over $\mathbf{F}_p$. [For the irreducibility: One approach - prove first that if α is a root then $\alpha + 1$ is also a root. Another approach - suppose it's reducible and compute derivatives.]

2. Qualifying exam exercises

Exercise 9
Let $f(x) = x^2 + x + 2 \in \mathbf{F}_5[x]$.
(a) Prove that $f(x)$ is irreducible in $\mathbf{F}_5[x]$.
(b) Explain why $f(x)$ divides the polynomial $x^{25} - x$ in $\mathbf{F}_5[x]$.
(c) How many irreducible quadratic polynomials are there in $\mathbf{F}_5[x]$?

Exercise 10
Let $\overline{\mathbf{Q}} \subset \mathbf{C}$ be the subfield of elements that are algebraic over $\mathbf{Q}$. Prove that the field extension $\mathbf{Q} \subset \overline{\mathbf{Q}}$ has infinite degree.

Exercise 11
Suppose that $\alpha, \beta \in \mathbf{C}$ have minimal polynomial over $\mathbf{Q}$ of degree 2 and 3, respectively. Can $\alpha + \beta$ have minimal polynomial over $\mathbf{Q}$ of degree 5? Give an example or prove that this is not possible.

Exercise 12
Construct a field with 32 elements. Prove that your construction produces a field with exactly 32 elements.