Math 230b: Algebra Linear Algebra Exam Problems

This is a list of recent problems from UCI qualifying and comprehensive/advisory exams that deal with linear algebra. This is not all of the linear algebra problems. For example, I have omitted the ones that deal with Hermitian operators and things like that. These problems are related to 230b material in the sense that many of them involve concepts like Jordan canonical form and minimal/characteristic polynomials of matrices.

1. Let X be a 3×3 complex matrix. Find all solutions of the equation $X^2 - X = 0$ up to similarity. Use the Jordan canonical form.
2. Let T be a 3×3 matrix with complex coefficients. Describe all possible solutions of the equation $T^3 = T$.
3. Find all 4×4 matrices A with real coefficients such that $A^3 = I$ where I is the identity matrix.
4. Classify, up to conjugation, all 4×4 real matrices with minimal polynomial $(x^2 + 4)(x - 1)$.
5. Determine all real matrices A with characteristic polynomial $x^3(x^2 + 1)$ up to conjugation.
6. Let X be a 5×5 complex matrix. Find all solutions of the equation $X^2 - X = 0$.
7. Find two 4×4 matrices with the same characteristic and minimal polynomials that are not similar. (Fully justify your answer.)
8. Prove that similar matrices A and B have the same minimal polynomial.
9. Find two matrices having the same characteristic polynomials and minimal polynomials but different Jordan canonical forms. Fully justify your answer.
10. Classify (up to conjugation by an orthogonal matrix) all matrices of size 4×4 with real coefficients which are orthogonal and skew symmetric ($A^t = -A$) at the same time.
11. (a) Give an example of a square matrix A with rational coefficients having characteristic polynomial $X^5 - X^3$ and minimal polynomial $X^4 - X^2$.
    (b) Suppose that A and B are square matrices with coefficients in a field k and both A and B have characteristic polynomial $X^5 - X^3$ and minimal polynomial $X^4 - X^2$. Show that A and B are similar, that is, there is an invertible matrix P with coefficients in k such that $PAP^{-1} = B$.
12. Suppose that A, B are elements of $M_2(\mathbb{C})$ such that $A^2 = B^3 = I$, $ABA = B^{-1}$ with $A \neq I$, $B \neq I$. If $D \in M_2(\mathbb{C})$ commutes with A and B, show that D is a scalar matrix, that is, a scalar multiple of I.
13. The group $\mathrm{GL}_2(\mathbb{C})$ acts on the set $M_2(\mathbb{C})$ by conjugation. Classify the orbits of this action.
14. If V is a vector space and $V = A \oplus B = C \oplus D$ with $A \cong C$, does it follow that $B \cong D$? Justify your answer.

15. Are there complex $n \times n$ matrices A and B for which $AB - BA = I$? Justify your answer.

16. Let A and B be any $n \times n$ matrices over $\mathbb{C}$. Is it possible that $ABA - BAB = I$? Please explain.

17. Let A and B be two $n \times n$ complex matrices. Let I_n be the $n \times n$ identity matrix.

    (a) Assume that A is non-singular. Show that $\det(I_n - AB) = \det(I_n - BA)$.

    (b) Show that $\det(I_n - AB) = \det(I_n - BA)$ always holds, even if A and B are singular.

18. Let M be an $n \times n$ matrix over a field F.

    (a) If F has characteristic zero, show that M is nilpotent if and only if $\operatorname{tr}(M^i) = 0$ for $1 \leq i \leq n$.

    (b) Give an example to show that the same statement is not true if the field F has characteristic $p > 0$.

19. Let A be an $n \times n$ complex matrix such that $A^k = I$ for some k. Show that $|\operatorname{tr}(A)| \leq n$.

20. Suppose that A is a nilpotent matrix. Show that $\det(A + I) = 1$.

21. Let U, V and W be subspaces in a vector space. Is it always true that

$$U \cap (V + W) = U \cap V + U \cap W?$$

    Give a proof or a counterexample.

22. Let W_1 and W_2 be linear subspace of a vector space V with $\dim(W_1) = r$ and $\dim(V) = s$. What are the possible values of $\dim(W_1 \cap W_2)$? Fully justify your answer.

23. Let V be a finite dimensional vector space over $\mathbb{C}$. Suppose that $T : V \to V$ is linear and $p \in \mathbb{Q}[x]$.

    (a) Define $p(T)$.

    (b) Show that if λ is an eigenvalue of T, then $p(\lambda)$ is an eigenvalue of $p(T)$.

    (c) Show that if λ is an eigenvalue of $p(T)$, then there is an eigenvalue λ' of T such that $\lambda = p(\lambda')$.

24. Let V and W be vector spaces of dimensions m and n respectively, and let $T : V \to W$ be linear transformations between them. Suppose there is a non-zero element f of the dual space W^* of W such that the equation $w = Tv$ has a solution if and only if $f(w) = 0$. Find the dimension of the kernel of T.

25. Suppose that V is a vector space and let $\mathrm{GL}(V)$ be the group of all invertible linear transformations from V to itself. Suppose G is a subgroup of $\mathrm{GL}(V)$ and define R to be the set of all linear transformations $T : V \to V$ such that $T(g(v)) = g(T(v))$ for every $g \in G$ and $v \in V$.

    (a) Show that R is a ring.

    (b) Suppose further that if W is any subspace of V such that $g(W) \subset W$ for every $g \in G$, then either $W = 0$ or $W = V$. Prove that if $T \in R$ and T is not the zero transformation, then T is invertible and $T^{-1} \in R$. (Hint: If $T \in R$, what can you say about the kernel and image of T?)