Math 230c: Algebra Fields Exam Problems

This document contains a list of problems from recent UCI qualifying exams and comprehensive/advisory exams that involve material from early in 230c.

1. Find all primes $p > 2$ for which the polynomial $x^2 + x + 1$ is irreducible in $\mathbb{F}_p[x]$.
2. Let $p > 2$ be a prime number. Prove that $1 \cdot 2 \cdots (p-1) \equiv -1 \pmod{p}$.
3. Let $f(x) = x^2 + x + 2 \in \mathbb{F}_5[x]$.
   (a) Prove that $f(x)$ is irreducible in $\mathbb{F}_5[x]$.
   (b) Explain why $f(x)$ divides the polynomial $x^{25} - x$ in $\mathbb{F}_5[x]$.
   (c) How many irreducible quadratic polynomials are there in $\mathbb{F}_5[x]$?
4. What is the number of irreducible degree 4 polynomials in $\mathbb{F}_p[x]$, where p is a prime?
5. Construct a field with 32 elements. Prove that your construction produces a field with exactly 32 elements.
6. Determine the maximal ideals of the following rings and fully justify your answers:
   (a) $\mathbb{Q}[x]/\langle x^2 - 5x + 6\rangle$,
   (b) $\mathbb{Q}[x]/\langle x^2 + 4x + 6\rangle$.
7. Let F be a field such that the multiplicative group F^* is finitely generated. Show that F is finite. (Hint: Eliminate the case of characteristic zero by proving that $\mathbb{Q}^*$ is not finitely generated. If F has finite characteristic p and $x \in F^*$ is of infinite multiplicative order then note x cannot be algebraic over the finite subfield $\mathbb{F}_p \subset F$ and consider irreducibles in $\mathbb{F}_p[x]$.)
8. Let $\mathbb{F}_q$ denote the finite field with q elements, and let $\mathbb{F}_q^*$ denote the elements of $\mathbb{F}_q$ that have a multiplicative inverse. Suppose that p is prime and suppose r and N are positive integers. Consider the map $\mathbb{F}_{p^r}^* \to \mathbb{F}_{p^r}^*$ that sends x to x^N. What is the cardinality of its kernel and image? (Fully justify your answer.)
9. Suppose that E/F is a finite extension of the form $E = F(r)$ with $[E : F]$ odd. Prove that $E = F(r^2)$.
10. Suppose that E/F is a field extension of degree $[E : F] = p$ and that p is prime. Show that for any $a \in E$ we have either $F(a) = F$ or $F(a) = E$.
11. Let $F < E$ and $F < K$ be field extensions. Suppose $[E : F] = n$ and $[K : F] = m$ where m and n are relatively prime. Prove that $[EK : F] = mn$.
12. Let F and E be finite field with $F < E$. Prove that the cardinality of E is an integral power of the cardinality of F, that is, $|E| = |F|^k$ for some integer k.
13. Let $\overline{\mathbb{Q}} \subset \mathbb{C}$ be the subfield of elements that are algebraic over $\mathbb{Q}$. Prove that the field extension $\mathbb{Q} \subset \overline{\mathbb{Q}}$ has infinite degree.

14. If $F < E$ is a field extension of finite degree, prove that E is algebraic over F.

15. Let $F < E$ be a field extension. If $a, b \in E$ are algebraic over F show that $a + b$ is also algebraic over F.

16. Let F be a field. Show that a polynomial $f(x) \in F[x]$ has no multiple roots if and only if $f(x)$ and its derivative $f'(x)$ are relatively prime. (Hint: You may assume that f has a splitting field E containing F.)

17. Suppose that $\alpha, \beta \in \mathbb{C}$ have minimal polynomials over $\mathbb{Q}$ of degrees 2 and 3, respectively. Can $\alpha + \beta$ have minimal polynomiial over $\mathbb{Q}$ of degree 5? Give an example or prove that it is not possible.