Remark: the exercise below will be graded carefully. Give explanations and computations.

Exercise 1

(a) Let A be an $n \times n$ matrix which in invertible. Prove that $A^T A$ is invertible. Be sure to justify each step in your proof completely (2 points).

(b) (hard) Let A be an $m \times n$ matrix such that the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Prove that $A^T A$ is invertible (1 point).

Exercise 2

Let

$$A = \begin{bmatrix} 2 & 0 & 10 \\ 0 & 8+x & -3 \\ 0 & 4 & x+1 \end{bmatrix}.$$

(a) Find all values of x such that A is invertible. Make sure that you completely justify your answer (2 points).

(b) Compute the inverse of A when $x = -3$ (2 points).

(c) For all x such that A is not invertible, find all solutions of the equation $A\mathbf{x} = \mathbf{0}$. (2 points).

d) Compute A^2 when $x = 0$. (1 point).