3A: Extra exercises 6

Remark: the exercise below will be graded carefully. Give explanations and computations.

Exercise 1 ($5 = 3 + 2$ points)
(a) Prove the following statement. For $\mathbf{u}, \mathbf{v} \in \mathbf{R}^n$ one has:

$$|\mathbf{u} \cdot \mathbf{v}| \leq ||\mathbf{u}|| ||\mathbf{v}||.$$

Hint: Let $\mathbf{w} = \text{Proj}_{\mathbf{v}}(\mathbf{u})$. Expand $||\mathbf{u} - \mathbf{w}||^2 \geq 0$.
(b) Prove that for $a, b, c \in \mathbf{R}$ one has

$$2a^2 + 3b^2 + c^2 \leq \sqrt{2a^2 + 9b^2 + c^2}\sqrt{2a^2 + b^2 + c^2}.$$

Exercise 2 ($5 = 1 + 1 + 2 + 1$ points)
Let A be an $n \times n$ matrix such that $A^2 = A$.
(a) Show that 0 and 1 are the only eigenvalues of A.
(b) Show that $E_0 = \text{Nul}(A)$ and $E_1 = \text{Col}(A)$ (here E_i is the eigenspace at eigenvalue i).
(c) Show that any $\mathbf{u} \in \mathbf{R}^n$ can be written uniquely as $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$ where $\mathbf{u}_1 \in \text{Nul}(A)$ and $\mathbf{u}_2 \in \text{Col}(A)$.
(d) Give an example of such an A with $A^2 = A$ where the null space and column space are not orthogonal (in the book, we only study orthogonal projections, where this is the case).