230C: Homework 1

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1. Exercises from secret sources

Exercise 1
Let $K \subseteq L \subseteq M$ be a tower of algebraic extensions. Prove: $K \subseteq L$ and $L \subseteq M$ are separable if and only if $K \subseteq M$ is separable.

Exercise 2
Let $L = \mathbf{F}_p(S, T)$ be the field of rational functions in two variables over $\mathbf{F}_p$, and set $K = L^p = \{x^p : x \in L\}$.
(a) Prove: $K = \mathbf{F}_p(S^p, T^p)$, and $K \subseteq L$ is a field extension of degree p^2.
(b) Show that $K \subseteq L$ is not a primitive field extension.
(c) Give infinitely many different subfields E with $K \subseteq E \subseteq L$.

Exercise 3
Let L be a quadratic extension of a field K of characteristic not 2. Prove: $L \cong K(\sqrt{x}) \cong K[X]/(X^2 - x)$ for some $x \in K$. Show that we cannot remove the condition on the characteristic of L.

Exercise 4
Determine the degree over $\mathbf{Q}$ of the splitting fields of the following polynomials.
(a) $X^2 + X - 2$.
(b) $X^2 + 2X - 2$.
(c) $X^3 + 2X - 2$.
(d) $X^4 + 2X^2 + 2$.

Exercise 5
For a field K of characteristic $p > 0$ we call $[K : K^p]$ the imperfect degree of K (here $K^p = \{x^p : x \in K\}$). Prove:
(a) $[K : K^p] = p^{i(K)}$ with $i(K) \in \mathbf{Z}_{\geq 0} \cup \{\infty\}$.
(b) For every finite extension $K \subseteq L$ one has $i(L) = i(K)$.
(c) For every algebraic extension $K \subseteq L$ one has $i(L) \subseteq i(K)$.
(d) One has $i(K(T)) = i(K) + 1$.

2. Exercises from old qualifying exams

Exercise 6
True/False. For each of the following answer True or False and give a brief explanation.
(a) Every finied subgroup of $\mathrm{GL}_n(\mathbf{Q})$ is abelian.
(b) A finite extension of $\mathbf{Q}$ cannot have infinitely many distinct subfields.

Exercise 7
(a) What does it mean for a field to be perfect?
(b) Give an example of a perfect field. (No need to justify your answer.)
(c) Give an example of a field that is not perfect. (No need to justify your answer.)

Exercise 8
For each of the following, either give an exapmle or briefly explain why no cuh example exists:
(a) a quadratic extension of fields that is not separable.

(b) a nonabelian group in which all the proper subgroups are cyclic.
(c) an infinite field where every nonzero element has finite multiplicative order.
(d) a nonabelian group with trivial automorphism group.
(e) an element of order 4 in $\mathbf{R}/\mathbf{Z}$.

Exercise 9
Suppose F is a field of characteristic $p > 0$. Define a function $\phi : F \to F$ by $\phi(x) = x^p$.
(a) Show that ϕ is a field homomorphism.
(b) Show that if F is fintie, then ϕ is an automorphism.
(c) Give an example of a field F usch that ϕ is not an automorphism.

Exercise 10
Give an example of an extension of fields that is not separable. Compute its separable and inseparable degrees. (Fully justify your answers).