230C: Homework 2

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1. Exercises from secret sources

Exercise 1
Let $K \subseteq L$ be a quadratic extension of fields. Show that $K \subseteq L$ is normal.

Exercise 2
Let $K \subseteq L \subseteq M$ be a tower of finite extensions of fields. Prove or disprove the following statements.
(a) If $K \subseteq L$ and $L \subseteq M$ are normal, then $K \subseteq M$ is normal.
(b) If $K \subseteq M$ is normal, then $L \subseteq M$ is normal.
(c) If $K \subseteq M$ is normal, then $K \subseteq L$ is normal.

Exercise 3
Let K be a field and let $a \in K$.
(a) Let $n \in \mathbf{Z}_{>0}$. Let L be a finite extension of K which has an element α such that $\alpha^n = a$. Prove: there is a $b \in K$ with $a^{[L:K]} = b^n$ (hint: use the norm).
(b) Let p be a prime number. Prove: $f = X^p - a \in K[X]$ is irreducible in $K[X]$ if and only if f has no root in K.

Exercise 4
Let $L = \mathbf{Q}(X)$ be the field of rational functions over $\mathbf{Q}$ and let $\sigma \in \mathrm{Aut}(L)$ be the unique automorphism with $\sigma(X) = X + 1$. Prove that $G = \langle \sigma \rangle$ is an infinite subgroup of $\mathrm{Aut}(L)$, and that $L^G \subseteq L$ is not an algebraic extension. Also show that the map $H \mapsto L^H$ from the set of subgroups of G to the set of subfields of L is neither injective, nor surjective.

Exercise 5
Let $K \subseteq L$ be a Galois extension of degree n. Show that L is the splitting field of a polynomial $f \in K[X]$ of degree n. Is such an f necessarily irreducible?

Exercise 6
Let K be a field of characteristic not 2. Let $K \subseteq L$ be a quadratic extension.
(a) Prove: there is an $m \in K^* \setminus K^{*2}$ with $L = K(\sqrt{m})$, and the subgroup $\langle \overline{m} \rangle \subseteq K^*/K^{*2}$ is uniquely determined by L.
(b) Prove: there is a bijection between the set of quadratic extension of K (inside an algebraic closure $\overline{K}$) and the set of non-trivial elements of K^*/K^{*2} given by $L \mapsto (L^{*2} \cap K^*) \setminus K^{*2}$.

2. Exercises from old qualifying exams

Exercise 7
For each of the following, either give an example or state that non exists. In either case, give a brief explanation.
(a) An element $\alpha \in \mathbf{Q}(\sqrt{2}, i)$ such that $\mathbf{Q}(\alpha) = \mathbf{Q}(\sqrt{2}, i)$.
(b) A tower of field extensions $L \supseteq K' \supseteq K$ such that L/K' and K'/K are Galois extensions but L/K is not Galois.

Exercise 8
Suppose F is a field and $f(x) \in F[x]$ is irreducible. Suppose that E is the splitting field over F for $f(x)$, and that for some $\alpha \in E$, we have $f(\alpha) = f(\alpha+1) = 0$. Show

that the characteristic of F is not zero.

Exercise 9
Let $L/\mathbf{Q}$ denote a Galois extension with Galois group isomorphic to A_4.
(a) Does there exist a quadratic extension $K/\mathbf{Q}$ contained in L? Prove your answer?
(b) Does there exist a degree 4 polynomial in $\mathbf{Q}[x]$ with splitting field equal to L? Prove your answer.

Exercise 10
Consider the extension of fields $\mathbf{R}(T) \subset \mathbf{R}(T^{1/4})$, where T is an indeterminate.
(a) Is $\mathbf{R}(T^{1/4})/\mathbf{R}(T)$ Galois? Why or why not?
(b) Find all intermediate fields F such that $\mathbf{R}(T) \subseteq F \subseteq \mathbf{R}(T^{1/4})$, and prove that you have found all of them.