230C: Homework 4

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1. Exercises from secret sources

Exercise 1
Determine the degrees of the irreducible factors of $X^{13} - 1$ in $\mathbf{F}_5[X]$, $\mathbf{F}_{25}[X]$ and $\mathbf{F}_{125}[X]$.

Exercise 2
For which primes p is $\mathbf{F}_p[X]/(X^4 + 1)$ a field?

Exercise 3
Prove that for every prime p one of $2, 3, 6$ is a square modulo p. Conclude that the polynomials

$$X^6 - 11X^4 + 36X^2 - 36 = (X^2 - 2)(X^2 - 3)(X^2 - 6)$$

has a root modulo p for every prime p but no root in $\mathbf{Z}$.

Exercise 4 (Artin-Schreier-radicals)
Let K be a field of characteristic $p > 0$ and let $K \subseteq L$ be a cyclic extension of degree p. Prove: $L = K(\alpha)$ where α is a zero of an Artin-Schreier polynomial $f = X^p - X - a \in K[X]$. (hint: consider the resolvent $\sum_{i=0}^{p-1} i\sigma^i(x)$ where $x \in L$ has trace $1 = \mathrm{Tr}_{L/K}(x) = \sum_{i=0}^{p-1} \sigma^i(x)$ and $\langle \sigma \rangle = \mathrm{Gal}(L/K)$).

Exercise 5 (Dummit and Foote, 14.9.3)
Let p be an odd prime, let s and t be independent transcendentals over $\mathbf{F}_p$, and let F be the field $\mathbf{F}_p(s, t)$. Let β be a root of $x^2 - sx + t = 0$ and let α be a root of $x^p - \beta = 0$ (in some algebraic closure of F). Set $E = F(\beta)$ and $K = F(\alpha)$.
(a) Prove that E is a Galois extension of F of degree 2 and that K is a purely inseparable extension of E of degree p.
(b) Prove that K is not a normal extension of F. (if it were, conjugate β over F to show that K would contain a p-th root of s and then also a p-th root of t, so $[K : F] \geq p^2$, a contradiction).
(c) Prove that there is no field K_0 such that $F \subseteq K_0 \subseteq K$ with K_0/F purely inseparable and K/K_0 separable. (If there were such a field, use that purely inseparable extensions are normal and the fact that the composite of two normal extensions is again normal to show that K would be a normal extension of F).

2. Exercises from old qualifying exams

Exercise 6
Let p be a prime and F an algebraically closed field of characteristic p. Let $n = p^a m$, where m is a positive integer not divisible by p. How many n-th roots of unity are there in F? Prove your answer.

Exercise 7
Let K/F be an algebraic extension of fields and let R be a ring such that $F \subseteq R \subseteq K$. Prove that R is a field.

Exercise 8
Determine the splitting field of $x^5 - 2$ over the finite field $\mathbf{F}_3$. Then determine the

Galois group over $\mathbf{F}_3$ of $x^5 - 2$, both as an abstract group and as a set of automorphisms.

Exercise 9
Let n be a positive integer. Prove that the polynomial $f(x) = x^{2^n} + 8x + 13$ is irreducible over $\mathbf{Q}$.

Exercise 10
Let p be a prime. Prove that the Galois group for $x^p - 2$ over $\mathbf{Q}$ is isomorphic to the group of matrices

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

with $a, b \in \mathbf{F}_p$, $a \neq 0$.