230C: Homework 5

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1. Exercises from a secret source

Exercise 1 Let $n \geq 2$. Consider the standard representation V of S_n on $\mathbf{C}^n$, defined by $\sigma(e_i) = e_{\sigma(i)}$. Show that V is the sum of precisely 2 irreducible representations (hint: use Exercise 3; let S_n act on $\{1, 2, \ldots, n\}^2$; use Burnside's lemma).

2. Exercises from old qualifying exams

Exercise 2
Consider complex representations of the finite group G up to isomorphism.
(a) Show that if G is abelian, then every irreducible representation of G has degree 1.
(b) Show that the number of degree 1 representations of G is equal to $\#G/[G, G]$, where $[G, G]$ denotes the commutator subgroup of G.

Exercise 3
Let G be a finite group acting on a finite set S. Let $\mathbf{C}[S]$ be the abstract vector space over $\mathbf{C}$ with basis S. Let χ be the character of the corresponding representation of G on $\mathbf{C}[S]$.
(a) Show that for $\sigma \in G$, the value $\chi(\sigma)$ is the number of fixed points of σ in S.
(b) Show that the inner product $\langle \chi, 1_G \rangle$ is the number of G-orbits in S, where the inner product is given by $\langle \chi_1, \chi_2 \rangle = \frac{1}{|G|} \sum_{\sigma \in G} \chi_1(\sigma)\chi_2(\sigma^{-1})$.

Exercise 4
Let χ be the character of a d-dimensional complex representation ρ of a finite group G. Prove that $|\chi(g)| \leq d$ for all $g \in G$, and that if $|\chi(g)| = d$, then $\rho(g) = \zeta I$ for some root of unity ζ depending on g.

Exercise 5
Let $V \subset \mathbf{C}[X, Y, Z]$ be the 6-dimensional vector space of homogeneous polynomialds of degree 2 over $\mathbf{C}$. (A polynomial is homogeneous of degree 2 if it is a linear combination of monomials each of which has total degree 2, such as XZ or Y^2.). View V as a representation of S_3, withc S_3 acting by permuting the variables.
(a) Give the character table of S_3 (no proof required).
(b) What is the character of the representation of S_3 on V?
(c) Express the character of this representation as a sum of irreducible characters.

Exercise 6
Let $V = \mathbf{C}[S_3]$, the complex group ring of S_3. View V as a representation of S_3, with S_3 acting on V by conjugation (not by multiplication).
(a) Give the character table of S_3 (no proof required).
(b) What is the character of the representation of S_3 on V.
(c) Express the character of this representation as a sum of irreducible characters.

Exercise 7
Consider complex representations of the finite group S_4 up to isomorphism.
(a) Show that S_4 has exactly two one dimensional complex representations.
(b) Prove that its other pairwise non-isomorphic complex representations have dimension $2, 3, 3$.

Exercise 8
Give the character table (over $\mathbf{C}$) of the quaternion group Q_8. Justify your answer.

Exercise 9
Compute the character table of the dihedral group of order 8 and of order 10.

Exercise 10
Let $L_1, \ldots, L_r$ be all pairwise non-isomorphic complex irreducible representations of a group G of order 12. What are the possible values for their dimensions $n_i = \dim_{\mathbf{C}} L_i$? For each of the possible answers of the form $(n_1, \ldots, n_r)$ give an example of G which has such irreducible representations.