Math 2B: midterm

Friday August 19, 11:00 – 12:15pm

There are 8 exercises, worth a total of 100 points.
No electronic devices/books/notes allowed.
Provide computations and or explanations, unless stated otherwise.

Name:

Student ID:

Exercise 1 (5 points each) Integrate.

(a) $\int x^2 e^{-x}\,\mathrm{d}x$

(b) $\int \frac{x}{x^4+1}\,\mathrm{d}x$

(c) $\int \tan^5\theta \sec\theta\,\mathrm{d}\theta$

(d) $\int \left(\sin\theta\cos\theta + \frac{\cos\theta}{\sin\theta}\right)\mathrm{d}\theta$

(e) $\int \frac{1}{(1-x^2)^{3/2}}\,\mathrm{d}x$

Exercise 2 (5 points each) Determine whether each improper integral below is convergent or divergent. Evaluate those that are convergent.

(a) $\int_1^\infty \frac{1}{\sqrt{x-1}}\,\mathrm{d}x$

(b) $\int_e^\infty \frac{2}{x(\ln x)^2}\,\mathrm{d}x$

(c) $\int_2^\infty \frac{2}{z^2-1}\,\mathrm{d}z$.

Exercise 3 (5 points) Find $g'(x)$ if $g(x) = \int_x^{x^2} \frac{\sqrt[5]{t^2+1}}{\sin t + \cos t + e^{t^2}}\, \mathrm{d}t$.

Exercise 4 (5 points each) Consider the function $f(x) = x^2 - x$.
(a) Estimate, using Riemann sums, the area under $f(x)$ and above the x-axis for $x = 1$ to $x = 5$ using four rectangles and right-hand endpoints.

(b) Express $\int_1^5 f(x)\, \mathrm{d}x$ as a limit of Riemann sums.

(c) Evaluate the limit from b directly. One may use the formulas: $\sum_{i=1}^n i = \frac{n(n+1)}{2}$ and $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$.

Exercise 5

(a) (5 points) Find the area of the region bounded by $y = x$ and $y = \sqrt{x}$.

(b) (8 points) Find the volume of the solid generated by revolving the region described in part a about $y = 2$.

(c) (8 points) Find the volume of the solid whose basis is the region described in part a and whose cross-sections perpendicular to the y-axis are semi-circles with the square side in the xy-plane.

Exercise 6 (5 points) Find the average value of the function $f(x) = -3 + x^2 + \frac{1}{x}$ on $[1, 3]$.

Exercise 7 (8 points) Assume that f is a continuous function with $\int_0^3 f(x)dx = 3$ and $\int_1^0 f(x)dx = -1$. Compute $\int_0^1 (3f(2x+1) + 4)\, dx$.

Exercise 8 (6 points) A car moves in a straight line so that that its velocity at time t is $v(t) = t^2 - 4$. Find the *distance* travelled between $0 \leq t \leq 3$.

Solution 1:
(a) (integration by parts) $-x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C$.
(b) (substitution $u = x^2$) $1/2\tan^{-1}(x^2) + C$.
(c) (substitution $u = \sec\theta, du = \sec\theta\tan\theta$).

$$\begin{aligned}\int \tan^5\theta\sec\theta\,\mathrm{d}\theta &= \int (\sec^2\theta - 1)^2\tan\theta\sec\theta\,\mathrm{d}\theta \\ &= \int (u^2-1)^2\,\mathrm{d}u = \int 1 - 2u^2 + u^4\,\mathrm{d}u \\ &= u - 2/3u^3 + 1/5u^5 = \sec\theta - 2/3\sec\theta^3 + 1/5\sec\theta^5 + C.\end{aligned}$$

(d) (substitution $u = \sin\theta$) $1/2\sin^2\theta + \ln|\sin\theta| + C$.
(e) (inverse substitution) Set $x = \sin\theta$. The integral becomes $\int 1/\cos^2(\theta)\,\mathrm{d}\theta = \int \sec^2\theta\,\mathrm{d}\theta = \tan\theta + C = \frac{x}{\sqrt{1-x^2}} + C$.

Solution 2:
(a) $\int \frac{1}{\sqrt{x-1}}\,\mathrm{d}x = 2\sqrt{x-1}$, which goes to infinty as $x \to \infty$ and (and if the lower bound goes to 1): does not exist.
(b) $\int \frac{2}{x(\ln x)^2}\,\mathrm{d}x = -2(\ln x)^{-1}$ (substitute $u = \ln x$). The integral becomes 2.
(c) One has

$$1/(z^2-1) = 1/(z-1) - 1/(z+1).$$

The primitive is $\ln(z-1) - \ln(z+1)$. One has

$$\lim_{a\to\infty} \ln((z-1)/(z+1))]_2^a = -\ln 1/3 = \ln 3.$$

Solution 3:
Set $f(t) = \frac{\sqrt[5]{t^2+1}}{\sin t + \cos t + e^{t^2}}$. Then the answer is $2xf(x^2) - f(x)$.

Solution 4:
(a) $\Delta = 4/4 = 1$, and hence the esimate is $f(2)+f(3)+f(4)+f(5) = 2+6+12+20 = 40$.
(b)

$$\lim_{n\to\infty} 4/n\sum_{i=1}^{n} f(1+4i/n) = \lim_{n\to\infty} 4/n\sum_{i=1}^{n}\big((1+4i/n)^2 - (1+4i/n)\big)$$

(c) The limit is

$$\begin{aligned}&= \lim_{n\to\infty} 4/n\sum_{i=1}^{n} 4i/n + 16i^2/n^2 \\ &= \lim_{n\to\infty} 16/n^2 \cdot n(n+1)/2 + 64\cdot n(n+1)(2n+1)/6 = 16/2 + 64/3 = 88/3\end{aligned}$$

Solution 5:
(a) Intersection points are $(0,0)$ and $(1,1)$. One has

$$\int_0^1 (x^{1/2} - x)\,\mathrm{d}x = 2/3x^{3/2} - 1/2x^2|_0^1 = 2/3 - 1/2 = 1/6.$$

(b)

$$\begin{aligned}\pi\int_0^1 (2-x)^2 - (2-\sqrt{x})^2\,\mathrm{d}x =& \pi\int_0^1 x^2 - 5x + 4\sqrt{x}\,\mathrm{d}x \\ =& \pi(1/3x^3 - 5/2x^2 + 8/3x^{3/2})|_0^1 = 1/2\pi.\end{aligned}$$

(c) $A(y) = \pi/8(y-y^2)^2$. One has

$$\pi/8\int_0^1 (y-y^2)^2\,\mathrm{d}y = \pi/8(1/3y^3 - 2/4y^4 + 1/5y^5)|_0^1 = \pi/8(1/3 - 1/2 + 1/5) = \pi/240.$$

Solution 6:

$$\frac{1}{2}\int_1^3 f(x)\,\mathrm{d}x = \ln(3)/2 + 8/6.$$

Solution 7:
One has $\int_1^3 f(x)\,\mathrm{d}x = \int_1^0 f(x)\,\mathrm{d}x + \int_0^3 f(x)\,\mathrm{d}x = 2$. One then has ($u = 2x+1$, $du = 2dx$).

$$\int_0^1 3f(2x+1)+4\,\mathrm{d}x = 4 + 3\int_0^1 f(2x+1)\,\mathrm{d}x = 4 + 3/2\int_1^3 f(u)\,\mathrm{d}u = 4 + 3/2\cdot 2 = 7.$$

Solution 8:

$$\int_0^2 4 - t^2\,\mathrm{d}t + \int_2^3 t^2 - 4\,\mathrm{d}t = 23/3$$