Math 2B: midterm

Friday August 25 2017, 1:00 – 2:15pm

There are 9 exercises, worth a total of 100 points.
No electronic devices/books/notes allowed.
Provide computations and or explanations, unless stated otherwise.

Name:

Student ID:

Exercise 1 (5 points each) Integrate.

(a) $\int \frac{\sin(x)}{1+\cos^2(x)}\,\mathrm{d}x$

(b) $\int x^2 \sin(2x)\,\mathrm{d}x$

(c) $\int \tan^5(x)\sec^3(x)\,\mathrm{d}x$

(d) $\int \frac{\mathrm{d}x}{\sqrt{x^2+4}}$. You may use: $\int \sec(\theta)\,\mathrm{d}\theta = \ln|\sec(\theta)+\tan(\theta)| + C$.

Exercise 2 (5 points each) Determine whether each improper integral below is convergent or divergent. Evaluate those that are convergent.

(a) $\int_1^5 (x-1)^{-4/3}\,\mathrm{d}x$

(b) $\int_0^\infty xe^{-x}\,\mathrm{d}x$

(c) $\int_3^\infty \frac{1}{x^2-x-2}\,\mathrm{d}x$.

Exercise 3 (5 points each) Consider the function $f(x) = 2x$.
(a) Estimate, using Riemann sums, the area under $f(x)$ and above the x-axis for $x = 1$ to $x = 3$ using four rectangles and left-hand endpoints.

(b) Express $\int_1^3 f(x)\,\mathrm{d}x$ as a limit of Riemann sums.

(c) Evaluate the limit from part b directly (do not use the fundamental theorem of calculus). One may use the formula: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

Exercise 4 (5 points) Find $g'(x)$ if $g(x) = \int_x^{\sin(x)} \frac{t^2+2t}{e^t+1}\,dt$.

Exercise 5

(a) (5 points) Find the area of the region enclosed by $y = x$ and $y = x^4$.

(b) (10 points) Find the volume of the solid generated by revolving the region described in part (a) about $y = -1$.

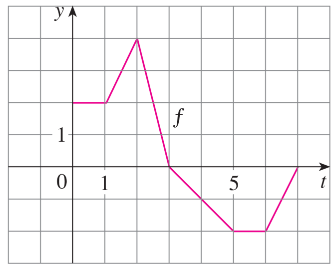
(c) (10 points) Find the volume of the solid whose base is the region described in part (a) and whose cross-sections perpendicular to the y-axis are squares with one side in the xy-plane.

Exercise 6 (5 points) Find the average value of the function $f(x) = \sin^3(x)$ on $[0, \pi]$.

Exercise 7 (5 points) Assume that f is a continuous function with $\int_1^4 f(x)\,\mathrm{d}x = 42$. Compute $\int_1^2 x f(x^2)\,\mathrm{d}x$.

Exercise 8 (5 points) A sail boat sails in a straight line so that that its velocity at time t is $v(t) = t^3 - 1$. Find the *distance traveled* of the boat between $t = 0$ and $t = 2$.

Exercise 9: (5 points) Let $g(x) = \int_1^x f(t)\,\mathrm{d}t$ where $f(t)$ is given by the following plot:



Fill in the blanks:

$$g(0) = \underline{\hspace{4cm}}$$
$$g(1) = \underline{\hspace{4cm}}$$
$$g(6) = \underline{\hspace{4cm}}$$
$$g'(2) = \underline{\hspace{4cm}}$$
$$g'(4) = \underline{\hspace{4cm}}.$$

Solutions:
1a: substitute $u = \cos(x)$. Then
$$\int \frac{\sin(x)}{1+\cos^2(x)}\,\mathrm{d}x = -\int \frac{1}{1+u^2}\,\mathrm{d}u = -\tan^{-1}(u) + C = -\tan^{-1}(\cos(x)) + C.$$
1b: integration by parts (standard computation)
$$1/4\cos(2x) - 1/2x^2\cos(2x) + 1/2x\sin(2x) + C.$$
1c: rewrite and substitute $u = \sec(x)$:
$$\begin{aligned}\int \tan^5(x)\sec^3(x)\,\mathrm{d}x &= \int (\sec^2(x)-1)^2\sec^2(x)\sec(x)\tan(x)\,\mathrm{d}x\\ &= \int (u^2-1)^2u^2\,\mathrm{d}u = 1/7u^7 - 2/5u^5 + 1/3u^3 + C\\ &= 1/7\sec^7(x) - 2/5\sec^5(x) + 1/3\sec^3(x) + C\end{aligned}$$
1d: $x = 2\tan(\theta)$ transforms the integral into $\int \sec(\theta)\,\mathrm{d}\theta$. Then apply the given formula to obtain
$$\int \frac{\mathrm{d}x}{\sqrt{x^2+4}} = \ln|\sqrt{x^2+4} + x| + C$$
2a: $-3(x-1)^{-1/3}|_1^5$, it diverges.
2b: $-xe^{-x} + e^{-x}|_0^\infty = 1$.
2c: Note: $1/(x^2-x-2) = 1/((x-2)(x+1)) = 1/3 \cdot 1/(x-2) - 1/3 \cdot 1/(x+1)$.
Hence integral is
$$\int_3^\infty 1/(x^2-x-2)\,\mathrm{d}x = 1/3\ln\left(\left|\frac{x-2}{x+1}\right|\right)|_3^\infty = \ln(4)/3.$$
This integral diverges.
3a: $1/2\left(f(1/2) + f(3/2) + f(2) + f(5/2)\right) = 1/2(2+3+4+5) = 7$.
3b: $\lim_{n\to\infty} 2/n\sum_{i=1}^n 2(1+2i/n)$ (also other options, this is right hand points).
3c: A simple computation shows that the limit becomes
$$\lim_{n\to\infty} 4 + 4(n+1)/n = 8.$$
4:
$$\cos(x)\frac{\sin(x)^2 + 2\sin(x)}{e^{\sin(x)}+1} - 3x^2\frac{x^2+2x}{e^x+1}.$$
5a: $\int_0^1 (x - x^4)\,\mathrm{d}x = 3/10$.
5b: $\int_0^1 \pi((x+1)^2 - (x^4+1)^2)\,\mathrm{d}x = 37/45\pi$.
5c: $\int_0^1 (y^{1/4} - y)^2\,\mathrm{d}y = 1/9$.
6:
$$1/\pi\int_0^\pi \sin^3(x)\,\mathrm{d}x = 1/\pi\int_0^\pi (1-\cos^2(x))\sin(x)\,\mathrm{d}x = \frac{1}{\pi}\cdot\left(-\cos(x) + 1/3\cos^3(x)\right)|_0^\pi = 4/3.$$
7: substitution:
$$\int_1^2 xf(x^2)du = \int_1^4 1/2f(u)du$$
$(u = x^2)$ and hence the result is $1/2 \cdot 42 = 21$.
8: $\int_0^1 (1-t^3)\,\mathrm{d}t + \int_1^2 (t^3-1)\,\mathrm{d}t = (t - t^4/4)|_0^1 + (t^4/4 - t)|_1^2 = 7/2$.
9: $g(0) = -2$, $g(1) = 0$, $g(6) = 1$, $g'(2) = 4$ and $g'(4) = -1$.