MATH 120A, FINAL

Friday, 9 December 2016

Name: ______________________________

Student ID: ______________________________

Instructions: This exam is 120 minutes. No notes, calculators, etc. are permitted. Clearly indicate your final answers, and show all work unless the problem explicitly says no justification is required. Cross out incorrect work. Do work in the space provided. Good luck.

Question	Score	Maximum
1		24
2		10
3		10
4		8
5		8
6		8
7		8
8		12
9		12
Total		100

1. (24 points) Give an example of each of the following, or briefly explain why no such example exists. (If you give an example, you don't need to explain why it works.)

   a. A group of order 10.

   b. An infinite group where the group operation is multiplication.

   c. A one-to-one homomorphism $\mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_{16}$.

d. An odd permutation $g \in S_n$, such that g^{-1} is an even permutation. (Reminder of terminology. A_n consists of all even permutations.)

e. An abelian group of order n containing no element of order n.

f. An element of order 3 in a group of even order.

2. (10 points) Let $G = \{2, 4, 6, 8\}$ with operation $\cdot_{10}$ of multiplication modulo 10. This is a group.

   a. What is the inverse of 2 in this group? Explain your answer.

   b. Give an isomorphism between G and a familiar group. You don't have to prove your map is an isomorphism, but state explicitly where each element in G maps.

3. (10 points) Let $G = \mathbb{Z}_4 \times \mathbb{Z}_4$, and let $H = \langle (1,2) \rangle$ be the cyclic subgroup generated by $(1,2)$.

   a. Write G as a disjoint union of left cosets of H. For each left coset, write down all the elements in that coset.

   b. Find elements $a, b, \ldots \in G$ so that the cosets in your disjoint union are equal to $[a], [b], \ldots$.

   c. Write down the table for the group operation on G/H using the symbols $[a], [b], \ldots$.

4. (8 points) Let $G = S_5$ and let H be the cyclic subgroup of G generated by the 4-cycle $(1\,5\,4\,2)$.
   a. List all the elements in H.

   b. Find an order 4 subgroup K of G such that $|H \cap K| = 2$, or prove that no such subgroup exists.

5. (8 points) Draw the subgroup diagram of the quaternion group Q_8. Label each vertex with the corresponding subgroup. You don't have to prove your answer. (Reminder. Every proper subgroup of Q_8 is cyclic.)

6. (8 points)

a. Give an example of a nontrivial homomorphism $\mathbb{Z}_3 \to U(\mathbb{Z}_{13})$. (You don't have to prove your function is a homomorphism.)

b. Let $\mathbb{Z}_{13} \rtimes \mathbb{Z}_3$ be the semidirect product of $\mathbb{Z}_{13}$ and $\mathbb{Z}_3$ with respect to the homomorphism you defined. Give an example of two elements $a, b \in \mathbb{Z}_{13} \rtimes \mathbb{Z}_3$ such that $ab \neq ba$. Write down the four elements a, b, ab, ba.

7. (8 points)

a. What is the identity element for the binary structure $(F, \circ)$, where F is the set of functions $\mathbb{R} \to \mathbb{R}$ and where $\circ$ denotes composition of functions?

b. What is the identity element for the binary structure $(F, \cdot)$, where F is the set of functions $\mathbb{R} \to \mathbb{R}$ and where $\cdot$ denotes pointwise multiplication?

c. Give an example of an element of F that is a unit in $(F, \circ)$ but not a unit in $(F, \cdot)$. (In other words, give an example of an element that has an inverse in the first binary structure but not in the second.)

8. (12 points) Let G be a group of order $3^2 5^2 = 225$. Prove that G contains an element of order 3 or an element of order 5 (or both). Do not use Cauchy's theorem or the Sylow theorems.

9. (12 points) Does there exist an integer n such that the symmetric group S_n is isomorphic to $D_4 \times \mathbb{Z}_3$? Prove your answer.