120A: Midterm

Wednesday August 23 2017, 8.00–9.40am

There are 6 exercises, worth a total of $100 = 20 + 14 + 20 + 12 + 10 + 24$ points.
No books, notes and calculators allowed.
Provide computations and or explanations, unless stated otherwise.

Name:

Student ID:

Exercise 1 (20 = 6 + 6 + 2 + 6 pts)
(a) Give the definition of a group.

(b) Consider the set $S = \mathbf{R} \setminus \{-1\}$ of real numbers except -1. Define $*$ on S by

$$a * b = a + b + ab.$$

Show that $(S, *)$ is a group.

(c) Show that the group in part b is abelian.

(d) Show that $(\mathbf{Q}, \cdot)$ where $\cdot$ is the standard multiplication on $\mathbf{Q}$ is not a group.

Exercise 2 (14 = 7 + 7 pts)
(a) Let $n \in \mathbf{Z}_{\geq 1}$. Show that $A = \left\{ \frac{a}{n} : a \in \mathbf{Z} \right\} \subset \mathbf{Q}$ is a subgroup of $(\mathbf{Q}, +)$.

(b) Find all the subgroups of $(\mathbf{Z}_{12}, +_{12})$ and construct the corresponding subgroup diagram.

Exercise 3 (20 = 4 + 4 + 4 + 4 + 4 pts)
Let $\sigma, \tau \in S_7$ be permutations defined by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 4 & 7 & 3 & 5 & 6 \end{pmatrix}, \quad \tau = (12)(376).$$

(a) Compute the disjoint cycle notation of σ.

(b) Compute σ^{-1}.

(c) Compute $\sigma\tau\sigma$.

(d) Compute τ^{32}.

(e) Is τ an even or an odd permutation?

Exercise 4 ($12 = 8 + 4$ pts)
(a) Let G be a group and let $g, h \in G$. Show that the equation $gx = h$ has a unique solution for $x \in G$.

(b) Given an example of a binary structure $(S, *)$ and $s, t \in S$ such that the equation $sx = t$ does not have a unique solution for $x \in S$.

Exercise 5 (10 pts)
Let G and H be isomorphic groups. Assume that G is a cyclic group. Show that H is a cyclic group.

Exercise 6 ($24 = 6 + 6 + 6 + 6$ pts)
True of false? Explain.
(a) The group S_5 has an element of order 7.

(b) Let G be a group and let $H_1 \subseteq G$ be a subgroup. Let $H_2 \subseteq H_1$ be a subgroup of H_1. Then H_2 is a subgroup of G.

(c) Let G be a group. Then there is a set A such that G is isomorphic with a subgroup of S_A (the permutation group of A).

(d) Let G be a cyclic group. Then there is a unique $a \in G$ with $G = \langle a \rangle$.