SOME SAMPLE FINAL QUESTIONS

1. Give the subgroup diagram of $\mathbb{Z}_2 \times \mathbb{Z}_6$. Label each vertex with the corresponding group.

2. a. Prove that if G is a finite group and $g \in G$ is an element, then the order of g divides the order of G.

   b. Prove or disprove: If G is a finite group and the positive integer n divides the order of G, then there exists an element in G of order n.

   c. Prove Fermat's Little Theorem: If p is a prime and if $a \in \mathbb{Z}$ is relatively prime to p, then $a^{p-1} \equiv 1 \bmod p$.

3. Prove that a group of order 6 contains an element of order 2. (Do not use Cauchy's theorem.)

4. Prove that the quaternion group Q_8 is not isomorphic to the dihedral group D_4.

5. Consider the binary operation defined on the set $\{a, b, c, d\}$ given in the following table.

$*$	a	b	c	d
a	b	d	a	c
b	d	c	b	a
c	a	b	c	d
d	c	a	d	b

   a. Is there an identity element for this operation? If so, what is it?

   b. What is the inverse of d for this operation? Briefly explain your answer.

   c. What is the order of a for this operation? Briefly explain your answer.

6. Let G be the group of bijections $\mathbb{Z} \to \mathbb{Z}$, with group operation composition of functions. (You don't need to prove that this is a group.)

   a. What is the identity element of this group?

   b. Give an example of an element of order 3 in this group.

   c. Give an example of an element of infinite order in this group.

   d. Is this group abelian?

7. Give an example or prove none exists.

   a. A one-to-one homomorphism $U(\mathbb{Z}_8) \to \mathbb{Z}_8$.

   b. A non-abelian subgroup of an abelian group.

   c. A non-cyclic infinite group.

   d. A normal subgroup H of $(\mathbb{R}, +)$ such that the factor group $\mathbb{R}/H$ has exactly two elements.

   e. A subgroup H of S_4 such that H is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and H is <u>not</u> a normal subgroup of S_4.

8. Answer True or False to the following and prove your answers.
   a. If H, K are groups, then $H \times K$ is isomorphic to $K \times H$.
   b. For every group G and element $g \in G$, there exists some subgroup H of G such that $g \in H$ and such that H is cyclic.
   c. If p is prime and G is a group of order p^3, then G has an element of order p. (Do not use Cauchy's theorem.)
   d. If n is an integer and G is a group of order n^3, then G has an element of order n. (Hint. Try $n = 4$.)