MATH 120B, FINAL

Wednesday, 14 June 2017

Name: ______________________________

Student ID: ___________________________

Instructions: This exam is 120 minutes. No notes, calculators, etc. are permitted. Clearly indicate your final answers, and show all work unless the problem explicitly says no justification is required. Cross out incorrect work. Do work in the space provided. Good luck.

Question	Score	Maximum
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		20
Total		100

1. (10 points)

   a. Consider the subset $\{0, 3, 6, 9, \ldots, 96, 99\} \subseteq \mathbb{Z}_{100}$. Is this a subring? Prove your answer.

   b. Consider the set $\mathbb{R}$ with the operations

$$a \oplus b := a + b \qquad \text{and} \qquad a \odot b := b.$$

   Is this a ring? Prove your answer.

2. (10 points) Prove or disprove: the rings $\mathbb{Z}[x]$ and $\mathbb{Q}[x]$ are isomorphic.

3. (10 points) How many ideals are in $\mathbb{Q} \times \mathbb{Q}$? Prove your answer.

4. (10 points)
   a. Prove or disprove: If R has unity and $N \subseteq R$ is an ideal, then R/N also has unity.

   b. Prove or disprove: If R is commutative and $N \subseteq R$ is an ideal, then R/N also is also commutative.

   c. Prove or disprove: If R is an integral domain and $N \subseteq R$ is an ideal, then R/N has no zero divisors.

5. (10 points) Prove or disprove: If R is an integral domain and $r, s \in R$ are elements such that $(r) = (s)$ (in other words, r and s generate the same principal ideal), then $s = ru$ for some unit $u \in R$.

6. (10 points)

a. Prove that $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{2})$ are not isomorphic. (You may use without proof that if $f : \mathbb{Q}(i) \to \mathbb{Q}(\sqrt{2})$ is an isomorphism, then $f(n) = n$ for all integers n.)

b. Prove or disprove: If F is a field and $f(x), g(x) \in F[x]$ are both irreducible and have the same degree, then the factor ring $F[x]/(f(x))$ is isomorphic to $F[x]/(g(x))$.

7. (10 points) Let F denote a field, let $a \in F$, and let $f(x) \in F[x]$ denote a polynomial such that $f(a) = 0$. Prove that if $\deg f(x) > 1$, then $f(x)$ is reducible.

8. (10 points) Consider the ring

$$\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}.$$

We define the function

$$N : \mathbb{Z}[\sqrt{2}] \to \mathbb{Z}, \qquad a + b\sqrt{2} \mapsto a^2 - 2b^2.$$

You may use without proof the following two facts:

- $N(rs) = N(r)N(s)$ for all $r, s \in \mathbb{Z}[\sqrt{2}]$
- $N(a + b\sqrt{2}) = (a + b\sqrt{2})(a - b\sqrt{2})$ where $a, b \in \mathbb{Z}$

a. Prove that $s \in \mathbb{Z}[\sqrt{2}]$ is a unit if and only if $N(s) = \pm 1$.

b. We can't use N as the function v in the definition of Euclidean domain. Why not? (Hint. Your answer should just be one sentence.)

9. (20 points) Give an example or briefly explain why no example exists.

   a. A subring of $M_{2\times 2}(\mathbb{R})$ which is commutative.

   b. A field $E \supseteq \mathbb{Q}(\sqrt{2})$ such that E has degree 11 over $\mathbb{Q}$.

(Continued on the next page.)

c. A field of characteristic 4.

d. A field with four elements.