MATH 120B, MAKE-UP FINAL EXAM

Monday, 12 June 2017

Name: ____________________________________

Student ID: ________________________________

Instructions: This exam is 120 minutes. No notes, calculators, etc. are permitted. Clearly indicate your final answers, and show all work unless the problem explicitly says no justification is required. Cross out incorrect work. Do work in the space provided. Good luck.

Question	Score	Maximum
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		20
Total		100

1. (10 points) Consider the ring $M_{2\times 2}(\mathbb{Z}_7)$.
   a. Does this ring have unity? Explain.

   b. Is this ring commutative? Prove your answer.

   c. What is the characteristic of this ring? Explain.

2. (10 points) Prove or disprove: the rings $2\mathbb{Z}$ and $3\mathbb{Z}$ are isomorphic.

3. (10 points) Are the following rings? Prove your answer.

   a. The set is $\mathbb{Z}$ and the operations are

$$a \oplus b := a + b$$
$$a \odot b := a + b$$

   b. The set is all subsets of $\mathbb{Z}$, and the operations are

$$S_1 \oplus S_2 := S_1 \cup S_2$$
$$S_1 \odot S_2 := S_1 \cap S_2$$

4. (10 points) Let R denote a subring of S and let $\varphi : R \to S$ be a ring homomorphism. Prove or disprove: The set
$$\{r \in R \mid \varphi(r) = r\}$$
is a ring.

5. (10 points)

   a. Define Euclidean domain.

   b. The following are Euclidean domains. For each of them, what is the function v? You don't need to explain or prove your answer.

      - $\mathbb{Z}$
      - $\mathbb{Z}[i]$
      - $\mathbb{C}[x]$

6. (10 points) Let $N \subseteq \mathbb{Q}[x]$ be the principal ideal generated by $x^5 - 10x^4 + 40$.

   a. Explain why $\mathbb{Q}[x]/N$ is a field.

   b. Find a basis for $\mathbb{Q}[x]/N$ as a vector space over the field $\mathbb{Q}$.

   c. Express the coset $x^6 + N$ as a $\mathbb{Q}$-linear combination of elements in your basis.

7. (10 points) Let R denote a ring with unity 1.
   a. Let $N \subseteq R$ be an ideal. Prove that the factor ring R/N has unity.

   b. Let $N \subseteq R$ be a proper ideal, and assume every non-zero element in R/N is a unit. Prove that N is a maximal ideal. (Don't assert that R/N is a field, because we don't know it's commutative.)

8. (10 points) Answer True or False to each of the following, and briefly explain your answers.
   a. True/False: The function $\varphi : \mathbb{R}[x] \to \mathbb{R}$ which sends a polynomial to the sum of its coefficients, i.e., which is given by
   $$\varphi : a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \mapsto a_n + a_{n-1} + \cdots + a_1 + a_0,$$
   is a ring homomorphism.

   b. True/False: Every field has exactly two ideals.

9. (20 points) Give an example or prove none exists.

a. A Hurwitz quaternion which is not in $\mathbb{Z}[i, j, k]$.

b. A prime ideal in $\mathbb{Z}_{12}$.

c. An element $\alpha \in \mathbb{C}$ such that the evaluation homomorphism $\phi_\alpha : \mathbb{Q}[x] \to \mathbb{C}$ is injective.

d. An element $\alpha \in \mathbb{Z}_{17}$ such that $(x - \alpha)$ is a factor of $x^{17} - x + 3$ in $\mathbb{Z}_{17}[x]$.