Math 120B
Sample final questions

1. Consider the set
$$\{(r_1, r_2, r_3, \dots) \mid r_i \in \mathbb{R}, |r_i| \leq |r_{i+1}|\},$$
equipped with coordinate-wise addition and multiplication. Does this structure form a ring? If so, is it commutative? Does it have unity? Is it an integral domain? Is it a field? What is its characteristic?

2. Define operations on $\mathbb{Z}$ by
$$\begin{aligned} a \oplus b &:= a + b \\ a \odot b &:= ab \text{ if } a, b \geq 0 \\ a \odot b &:= 0 \text{ if } a < 0 \text{ or } b < 0. \end{aligned}$$
Does this structure form a ring? If so, is it commutative? Does it have unity? Is it an integral domain? Is it a field? What is its characteristic?

3. For each of the following, give an example or prove none exists.
   a. A non-zero ring homomorphism $\mathbb{Z}_3 \to \mathbb{Z}_{15}$.
   b. A subring of $\mathbb{Z}_3[x]$ which is not an ideal in $\mathbb{Z}_3[x]$.
   c. A subring of $\mathbb{Z}$ which is not an ideal in $\mathbb{Z}$.
   d. An integral domain of characteristic zero in which $r \mapsto r^2$ is a ring homomorphism.
   e. A non-zero ring homomorphism from a field of characteristic 0 to a ring of characteristic 4.
   f. A prime ideal in $\mathbb{Q}[x]$ which is not a maximal ideal.
   g. A prime ideal in $\mathbb{Q} \times \mathbb{Q}$ which is not a maximal ideal.
   h. A non-empty subset of a ring which is closed under addition and multiplication which is not a subring. (Hint. Look in a ring without unity.)
   i. A finite ring without unity.
   j. A finite, non-commutative ring.
   k. An ideal which is not a principal ideal.
   l. Three pairwise non-isomorphic rings R_1, R_2, R_3, such that each has exactly four elements.
   m. A reducible polynomial over a field F that has no zeros in F.
   n. A field F in which $1 + 1 + 1 + 1 + 1 + 1$ is not a unit.

4. Prove that if $\gcd(a, 23) = 1$, then $a^{11} = \pm 1 \mod 23$.

5. Prove carefully that the rings $\mathbb{R}[x]$ and $\mathbb{C}[x]$ are not isomorphic.

6. Let $f(x) \in \mathbb{Q}[x]$ be a degree 2 polynomial, and let $N \subseteq \mathbb{Q}[x]$ be the principal ideal
$$N = (5x^{10} - 100x^5 + 10) \subseteq \mathbb{Q}[x].$$
True/False: $f(x) + N$ is a unit in $\mathbb{Q}[x]/N$.

7. Let $n \geq 1$ be a positive integer. Prove there exists K, an extension field of $\mathbb{Q}$, such that the degree of K over $\mathbb{Q}$ equals n. (Why doesn't your proof work for extensions of $\mathbb{Z}_p$?)

8. a. What is meant by the notation $\mathbb{Q}(\sqrt[4]{2}, i)$?

   b. What is the degree of $\mathbb{Q}(\sqrt[4]{2}, i)$ over $\mathbb{Q}$? (Hint. Use that if the degree of $K/\mathbb{Q}$ is m and the degree of K_1/K is n, then the degree of $K_1/\mathbb{Q}$ is mn.)

   c. Write out a basis for the $\mathbb{Q}$-vector space $\mathbb{Q}(\sqrt[4]{2}, i)$.

9. a. Define what it means for an element in an integral domain to be <u>irreducible</u>.

   b. Give an example of a prime $p \in \mathbb{Z}$ such that $p > 20$ and p is reducible in $\mathbb{Z}[i]$.

10. a. Define <u>Euclidean domain</u>.

    b. Define <u>principal ideal</u>.

    c. Prove that every ideal in a Euclidean domain is principal.

11. Let $\mathcal{O}$ denote the ring of Hurwitz integers. Prove that $\alpha \in \mathcal{O}$ is a unit if and only if $N(\alpha) = 1$. (You may use any other facts in your proof, but clearly state those facts when you use them.)