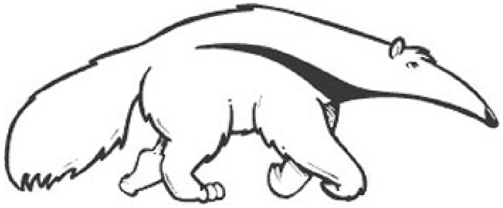
Constraint satisfaction problems

CS171, Fall 2017

Introduction to Artificial Intelligence

Prof. Richard Lathrop



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Constraint Satisfaction Problems

- What is a CSP?
  - Finite set of variables, $X_1, X_2, \ldots, X_n$
  - Nonempty domain of possible values for each: $D_1, \ldots, D_n$
  - Finite set of constraints, $C_1, \ldots, C_m$
    - Each constraint C_i limits the values that variables can take, e.g., $X_1 \neq X_2$
  - Each constraint C_i is a pair: C_i = (*scope*, *relation*)
    - Scope = tuple of variables that participate in the constraint
    - Relation = list of allowed combinations of variables

      May be an explicit list of allowed combinations

      May be an abstract relation allowing membership testing & listing

- CSP benefits
  - Standard representation pattern
  - Generic goal and successor functions
  - Generic heuristics (no domain-specific expertise required)

Example: Sudoku

- Problem specification

Variables: {A1, A2, A3, … I7, I8, I9}

Domains: D_i = { 1, 2, 3, … , 9 }

Constraints:

each row, column “all different”

alldiff(A1,A2,A3…,A9), ...

each 3x3 block “all different”

alldiff(G7,G8,G9,H7,…I9), ...

	1	2	3	4	5	6	7	8	9
A			2	4		6			
B	8	6	5	1			2		
C		1				8	6		9
D	9				4		8	6	
E		4	7				1	9	
F		5	8		6				3
G	4		6	9				7	
H			9			4	5	8	1
I				3		2	9		

Task: solve (complete a partial solution)

check “well-posed”: exactly one solution?

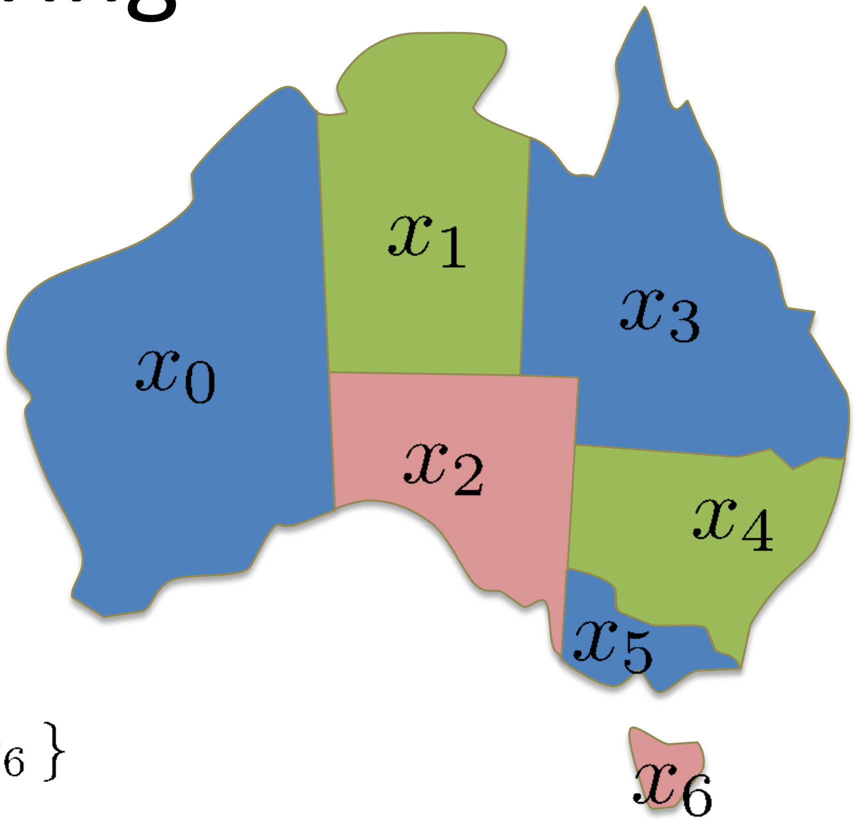
CSPs: What is a Solution?

- State: assignment of values to some or all variables
  - Assignment is complete when every variable has an assigned value
  - Assignment is partial when one or more variables have no assigned value
- Consistent assignment
  - An assignment that does not violate any constraint
- A solution to a CSP is a complete and consistent assignment
  - All variables are assigned, and no constraints are violated

- CSPs may require a solution that maximizes an objective function
  - Linear objective => linear programming or integer linear programming
  - Ex: “Weighted” CSPs

- Examples of applications
  - Scheduling the time of observations on the Hubble Space Telescope
  - Airline schedules
  - Cryptography
  - Computer vision, image interpretation

Example: Map Coloring



Variables: $\{ x_0 , x_1 , x_2 , x_3 , x_4 , x_5 , x_6 \}$

Domains: D_i = { red, green, blue }

Constraints: bordering regions must have different colors:

$$x_0 \neq x_1, \; x_0 \neq x_2, \; x_1 \neq x_2, \ldots$$

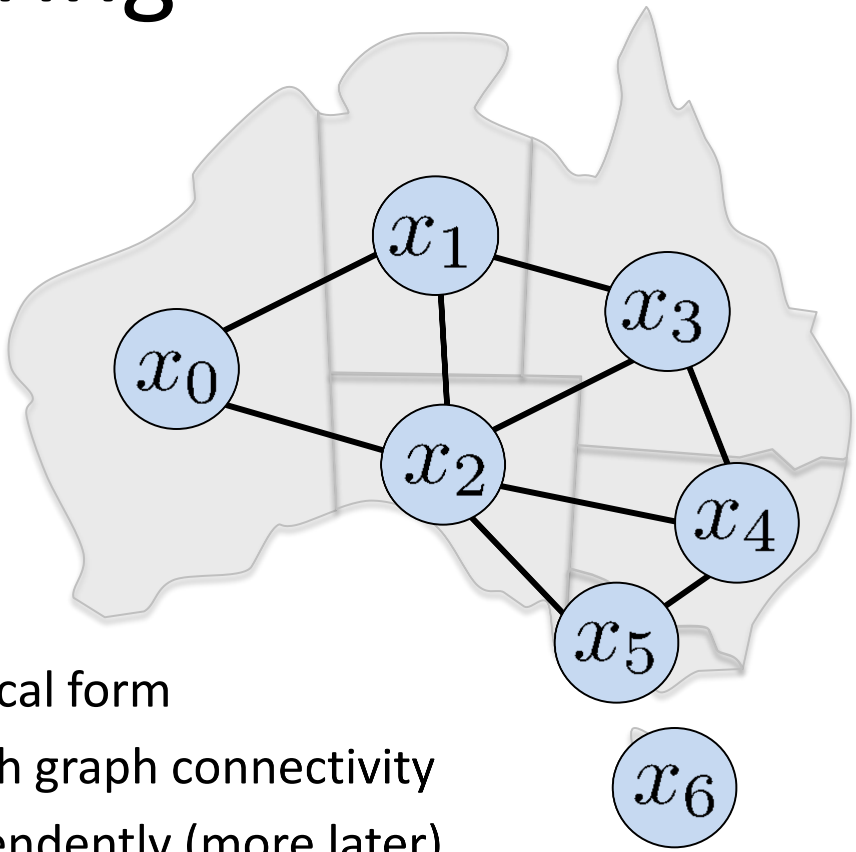
A ***solution*** is any setting of the variables that satisfies all the constraints, e.g.,

$$x_0 = blue, \; x_1 = green, \; x_2 = red, \; x_3 = blue,$$

$$x_4 = green, \; x_5 = blue, \; x_6 = red$$

Example: Map Coloring

- Constraint graph
  - Vertices: variables
  - Edges: constraints
    (connect involved variables)

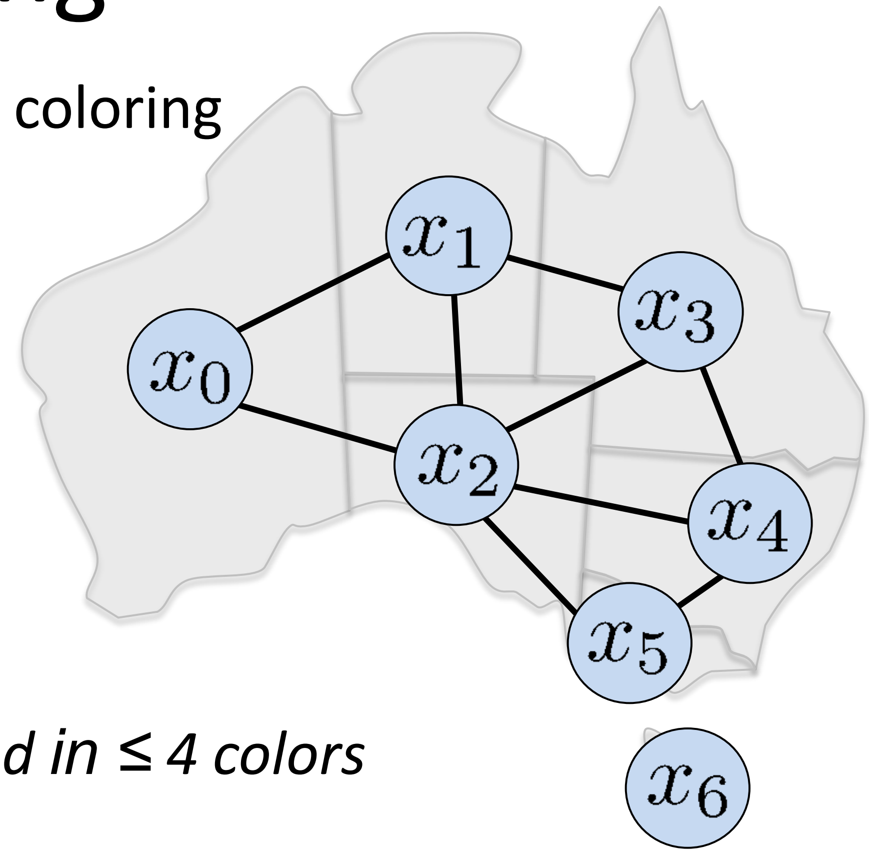


- Graphical model
  - Abstracts the problem to a canonical form
  - Can reason about problem through graph connectivity
  - Ex: Tasmania can be solved independently (more later)

- Binary CSP
  - Constraints involve at most two variables
  - Sometimes called “pairwise”

Aside: Graph coloring

- More general problem than map coloring
- Planar graph:
  graph in 2D plane with no edge crossings
- Guthrie's conjecture (1852)
  Every planar graph can be colored in ≤ 4 colors
- Proved (using a computer) in 1977 (Appel & Haken 1977)



Varieties of CSPs

- Discrete variables
  - Finite domains, size d => $O(d^n)$ complete assignments
    - Ex: Boolean CSPs: Boolean satisfiability (NP-complete)
  - Infinite domains (integers, strings, etc.)
    - Ex: Job scheduling, variables are start/end days for each job
    - Need a constraint language, e.g., $StartJob\_1 + 5 < StartJob\_3$
    - Infinitely many solutions
    - Linear constraints: solvable
    - Nonlinear: no general algorithm
- Continuous variables
  - Ex: Building an airline schedule or class schedule
  - Linear constraints: solvable in polynomial time by LP methods

Varieties of constraints

- Unary constraints involve a single variable,
  - e.g., SA ≠ green
- Binary constraints involve pairs of variables,
  - e.g., SA ≠ WA
- Higher-order constraints involve 3 or more variables,
  - Ex: jobs A,B,C cannot all be run at the same time
  - Can always be expressed using multiple binary constraints
- Preference (soft constraints)
  - Ex: “red is better than green” can often be represented by a cost for each variable assignment
  - Combines optimization with CSPs

Simplify…

- We restrict attention to:

- Discrete & finite domains
  - Variables have a discrete, finite set of values
- No objective function
  - Any complete & consistent solution is OK
- Solution
  - Find a complete & consistent assignment

- Example: Sudoku puzzles

Binary CSPs

CSPs only need binary constraints!

- Unary constraints
  - Just delete values from the variable's domain
- Higher order (3 or more variables): reduce to binary
  - Simple example: 3 variables X,Y,Z
  - Domains Dx={1,2,3}, Dy={1,2,3}, Dz={1,2,3}
  - Constraint C[X,Y,Z] = {X+Y=Z} = {(1,1,2),(1,2,3),(2,1,3)}

    (Plus other variables & constraints elsewhere in the CSP)

  - Create a new variable W, taking values as triples (3-tuples)
  - Domain of W is Dw={(1,1,2),(1,2,3),(2,1,3)}
    - Dw is exactly the tuples that satisfy the higher-order constraint
  - Create three new constraints:
    - C[X,W] = { [1,(1,1,2)], [1,(1,2,3)], [2,(2,1,3) }
    - C[Y,W] = { [1,(1,1,2)], [2,(1,2,3)], [1,(2,1,3) }
    - C[Z,W] = { [2,(1,1,2)], [3,(1,2,3)], [3,(2,1,3) }

    Other constraints elsewhere involving X,Y,Z are unaffected

Example: Cryptarithmetic problems

- Find numeric substitutions that make an equation hold:

$$
\begin{array}{ccccc}
 & & T & W & O \\
+ & & T & W & O \\
\hline
= & F & O & U & R
\end{array}
$$

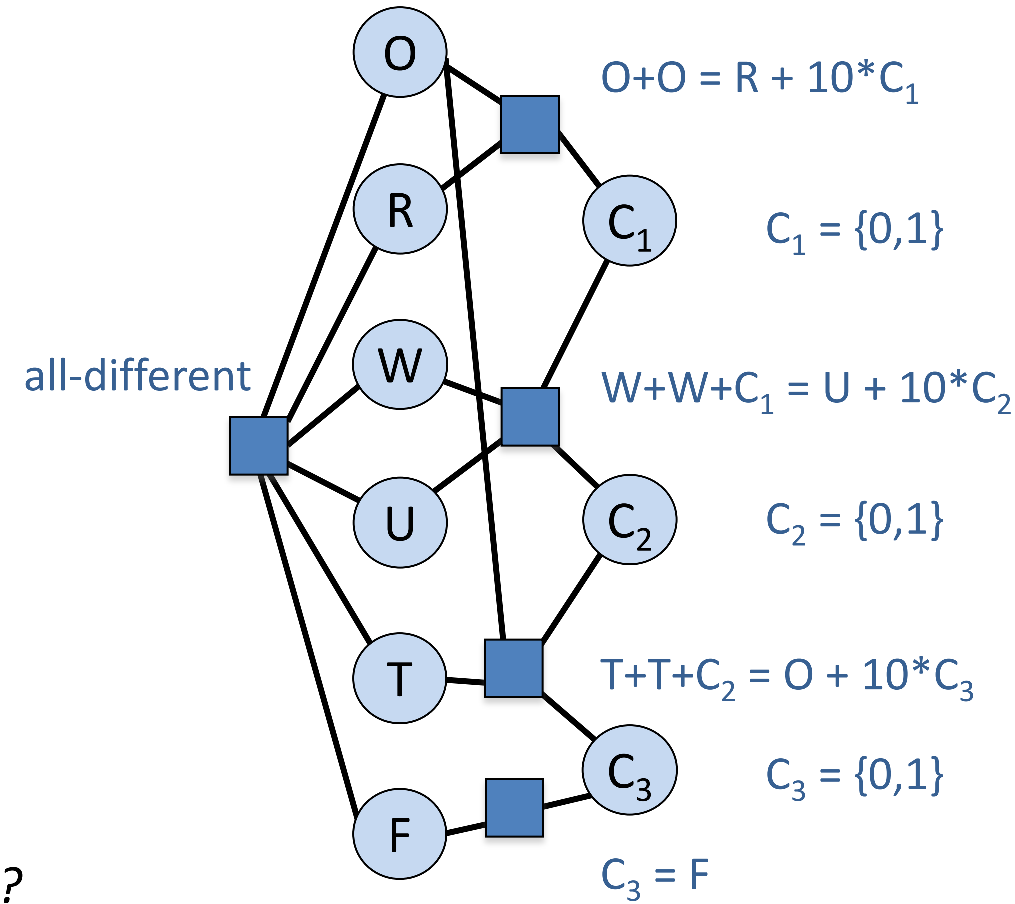
For example:

O = 4
R = 8
W = 3
U = 6
T = 7
F = 1

$$
\begin{array}{ccccc}
 & & 7 & 3 & 4 \\
+ & & 7 & 3 & 4 \\
\hline
= & 1 & 4 & 6 & 8
\end{array}
$$

Note: not unique – how many solutions?

Non-pairwise CSP:



Example: Cryptarithmetic problems

- Try it yourself at home:

```
      S E N D
+     M O R E
-------------
=   M O N E Y
```

(a frequent request from college students to parents)

(adapted from http://www.unitime.org/csp.php)

Random binary CSPs

- A random binary CSP is defined by a four-tuple (n, d, p_1, p_2)
  - n = the number of variables.
  - d = the domain size of each variable.
  - p_1 = probability a constraint exists between two variables.
  - p_2 = probability a pair of values in the domains of two variables connected by a constraint is incompatible.
    - Note that R&N lists compatible pairs of values instead.
    - Equivalent formulations; just take the set complement.
- (n, d, p_1, p_2) generate random binary constraints
- The so-called "model B" of Random CSP (n, d, n_1, n_2)
  - n1 = p_1 n(n-1)/2 pairs of variables are randomly and uniformly selected and binary constraints are posted between them.
  - For each constraint, $n_2 = p_2$ d^2 randomly and uniformly selected pairs of values are picked as incompatible.
- The random CSP as an optimization problem (minCSP).
  - Goal is to minimize the total sum of values for all variables.

CSP as a standard search problem

- A CSP can easily be expressed as a standard search problem.

- Incremental formulation
  - *Initial State*: the empty assignment {}
  - *Actions*: Assign a value to an unassigned variable provided that it does not violate a constraint
  - *Goal test*: the current assignment is complete

    (by construction it is consistent)
  - *Path cost*: constant cost for every step (not really relevant)

> **BUT:** solution is at depth n (# of variables)
> For BFS: branching factor at top level is *nd*
> next level: *(n-1)d*
> …
> Total: $n!\, d^n$ leaves! But there are only d^n complete assignments!

- Aside: can also use complete-state formulation
  - Local search techniques (Chapter 4) tend to work well

Commutativity

- CSPs are commutative.
  - Order of any given set of actions has no effect on the outcome.
  - Example: choose colors for Australian territories, one at a time.
    - **[WA=red then NT=green]** same as **[NT=green then WA=red]**

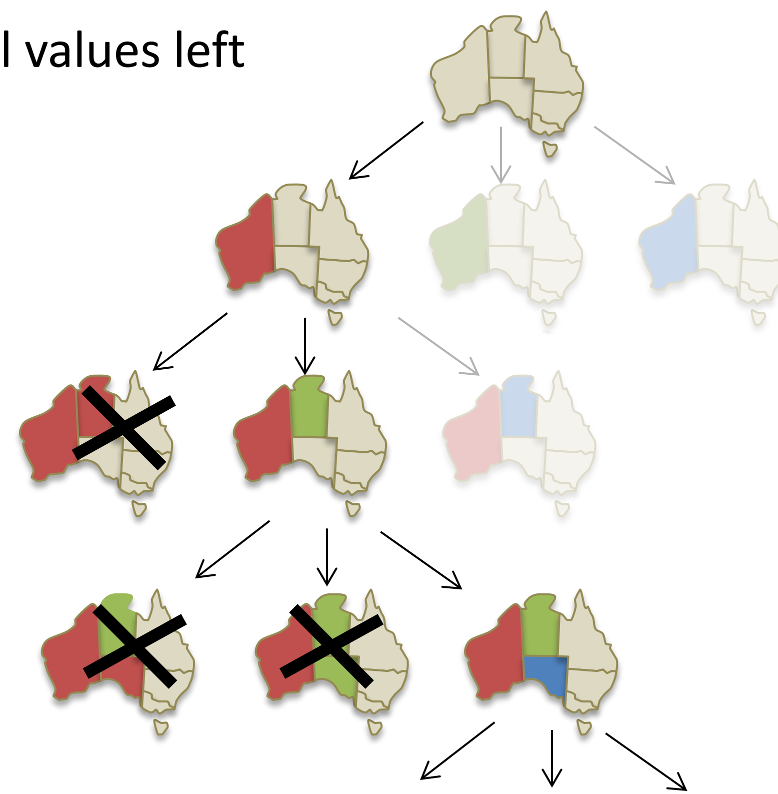
- All CSP search algorithms can generate successors by considering assignments for only a single variable at each node in the search tree

  $\Rightarrow$ there are d^n irredundant leaves

- (Figure out later to which variable to assign which value.)

Backtracking search

- Similar to depth-first search
  - At each level, pick a single variable to expand
  - Iterate over the domain values of that variable
- Generate children one at a time, one per value
  - Backtrack when a variable has no legal values left
- Uninformed algorithm
  - Poor general performance



Backtracking search

(R&N Fig. 6.5)

function BACKTRACKING-SEARCH(*csp*) **return** a solution or failure
  return RECURSIVE-BACKTRACKING(*{}* , *csp*)

function RECURSIVE-BACKTRACKING(*assignment, csp*) **return** a solution or failure
  if *assignment* is complete **then return** *assignment*
  var ← **SELECT-UNASSIGNED-VARIABLE**(VARIABLES[*csp*],*assignment*,*csp*)
  for each *value* **in** **ORDER-DOMAIN-VALUES**(*var, assignment, csp*) **do**
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    add *{var=value}* to assignment
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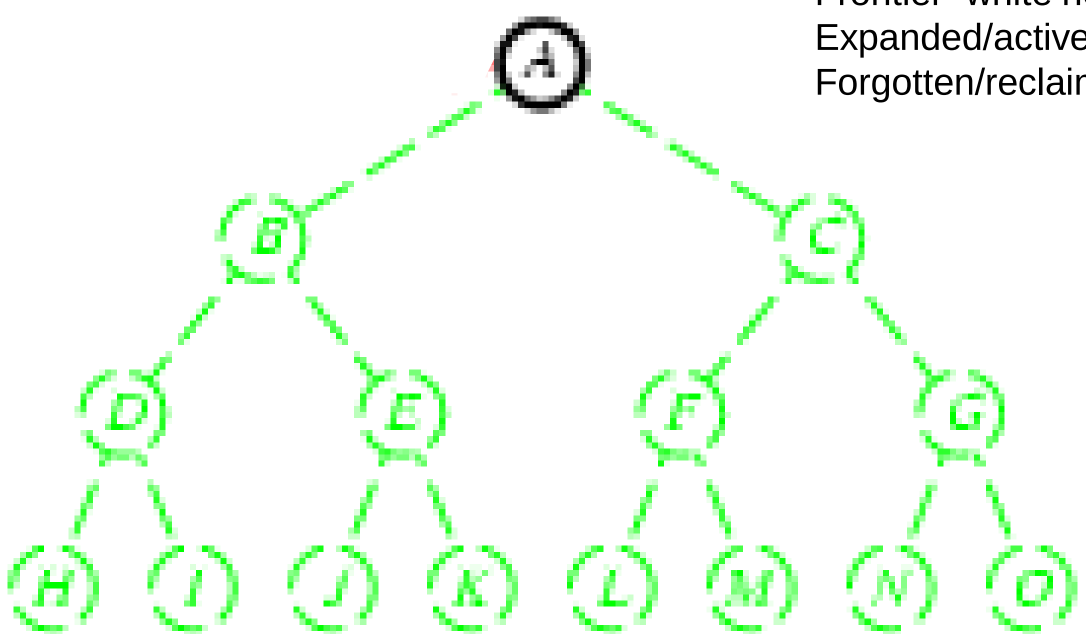
Backtracking search

- Expand *deepest* unexpanded node
- Generate ***only one*** child at a time.
- *Goal-Test* when inserted.
  - For CSP, Goal-test at bottom

Future= green dotted circles
Frontier=white nodes
Expanded/active=gray nodes
Forgotten/reclaimed= black nodes



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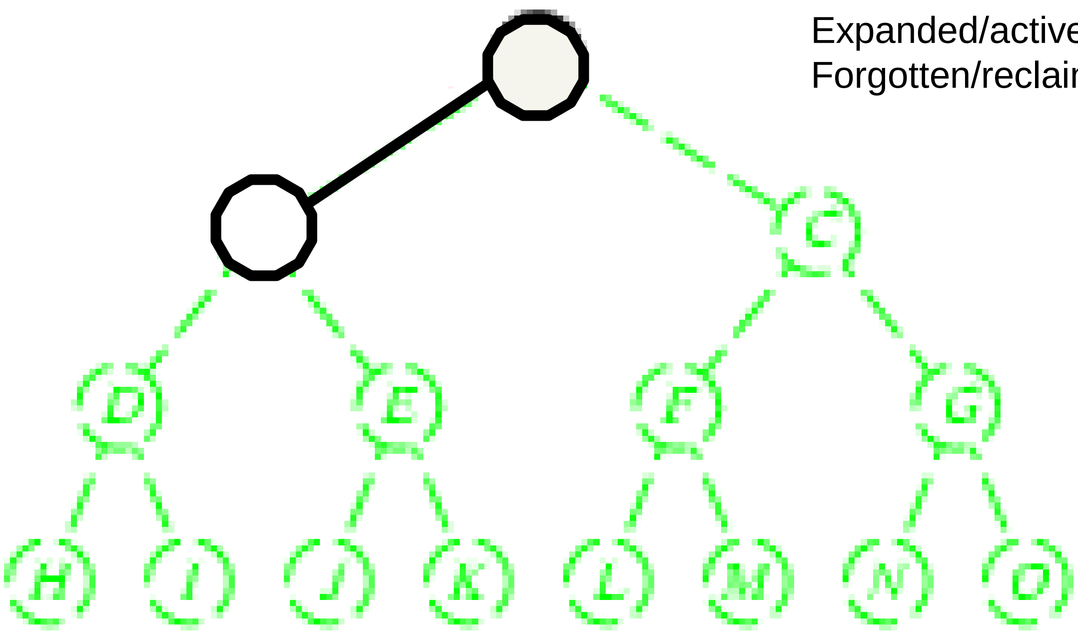

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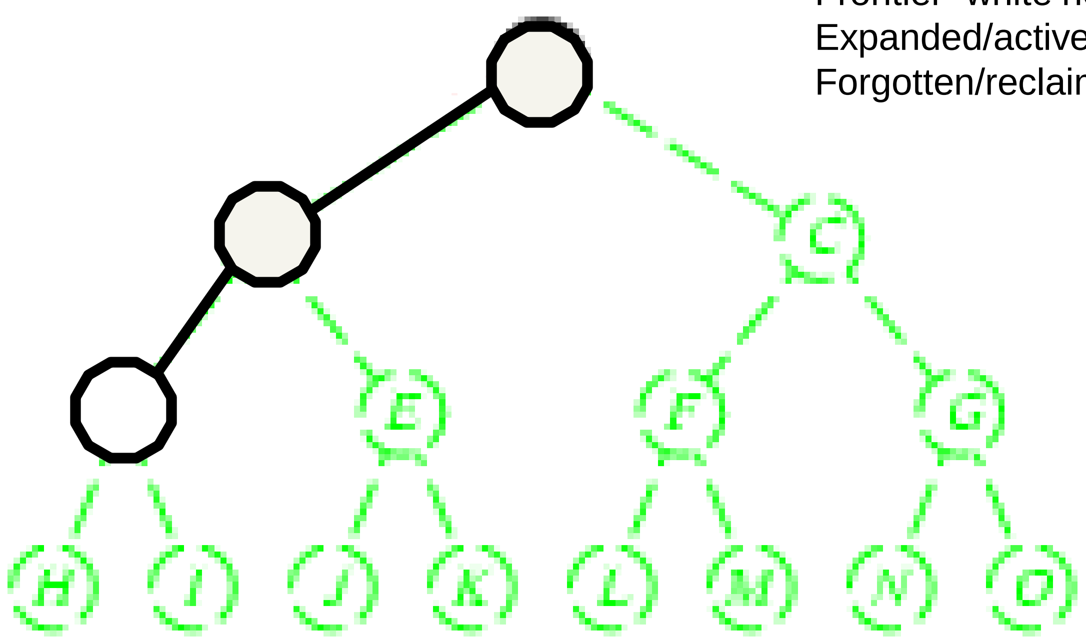

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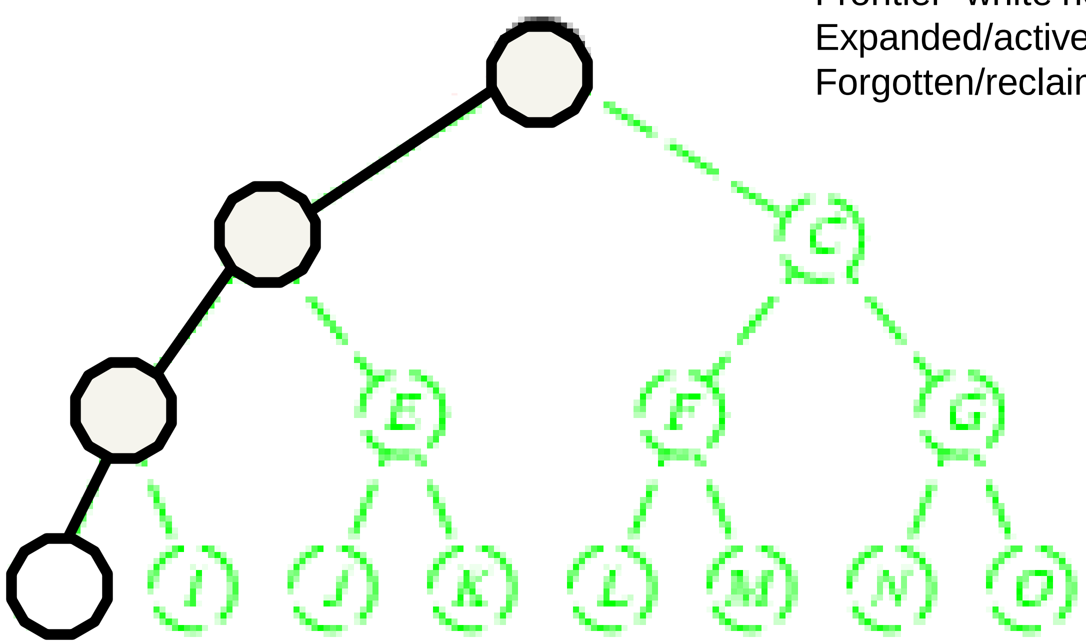

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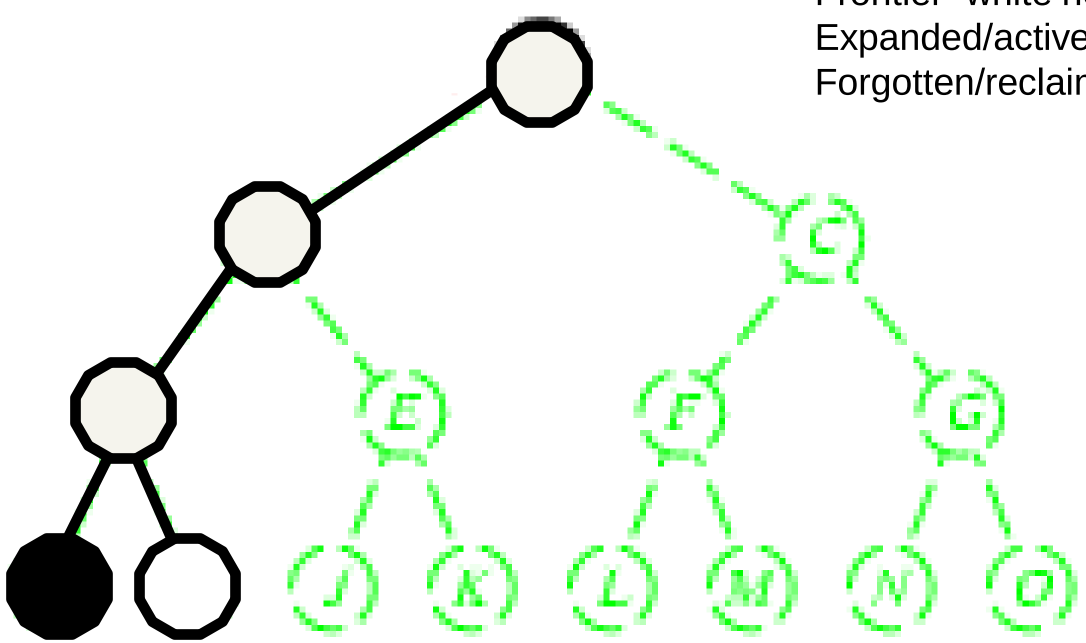

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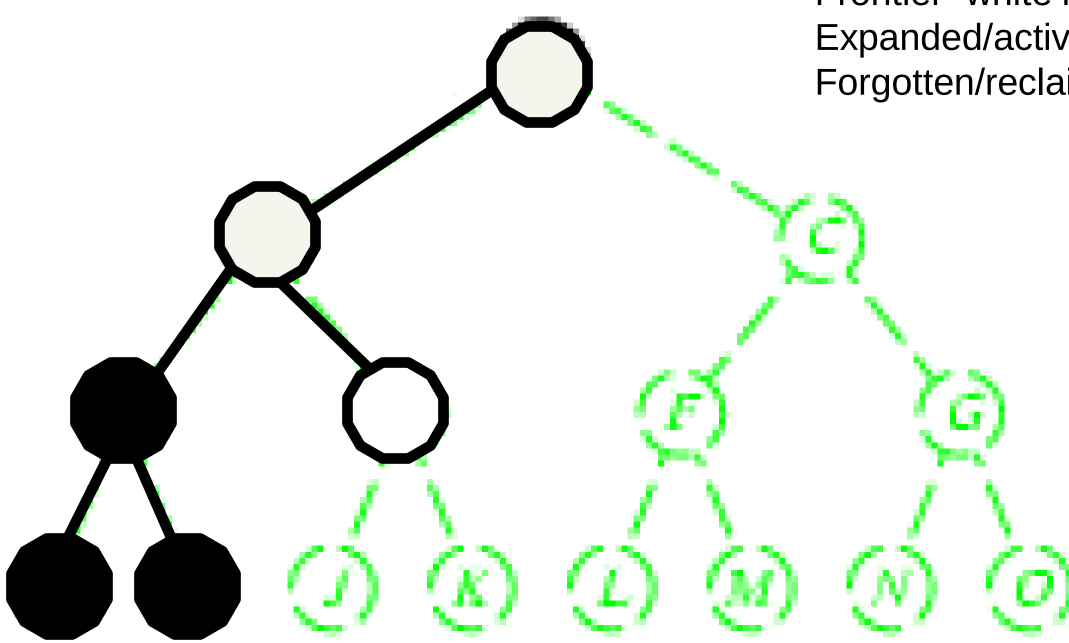

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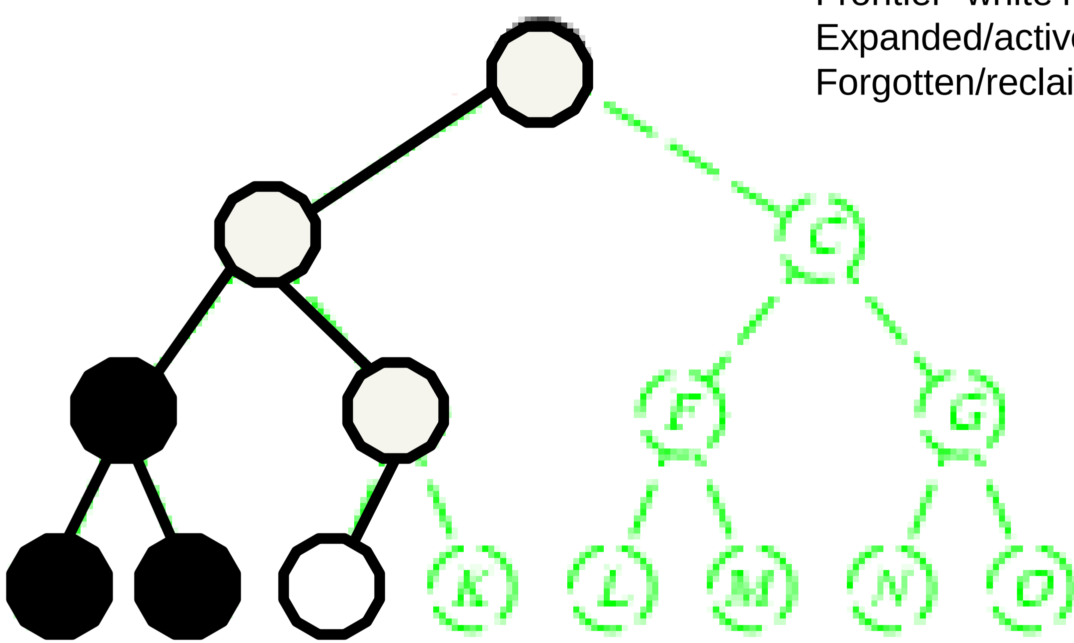

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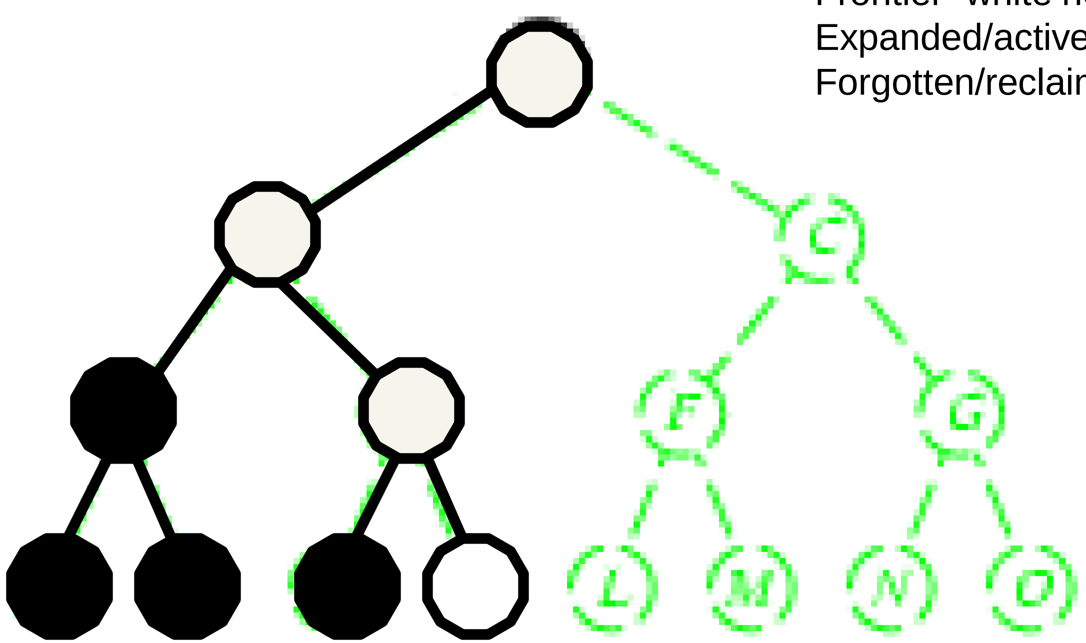

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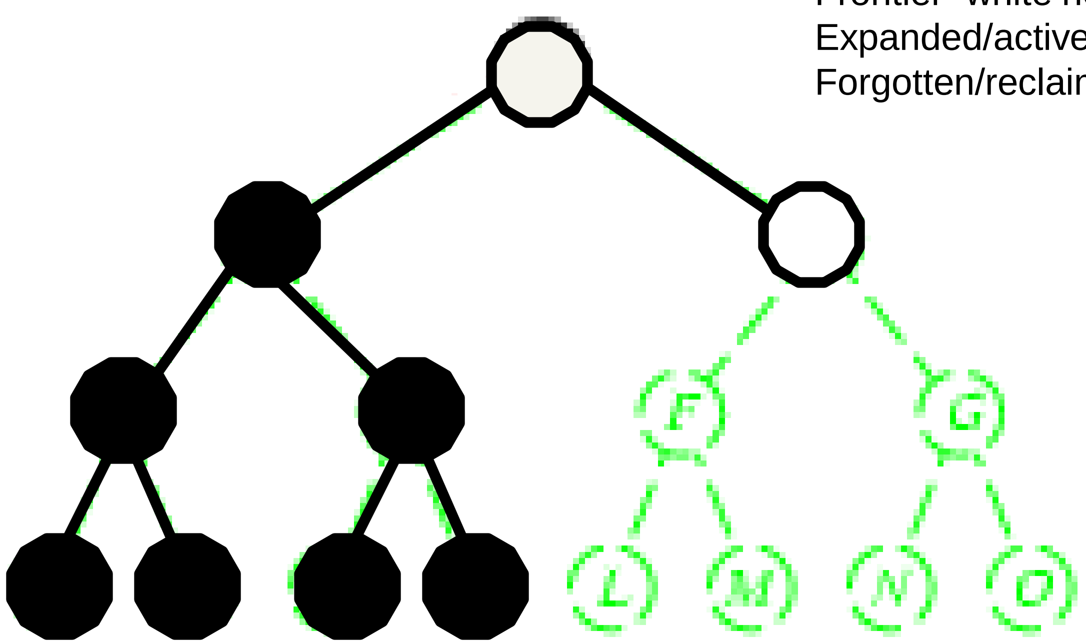

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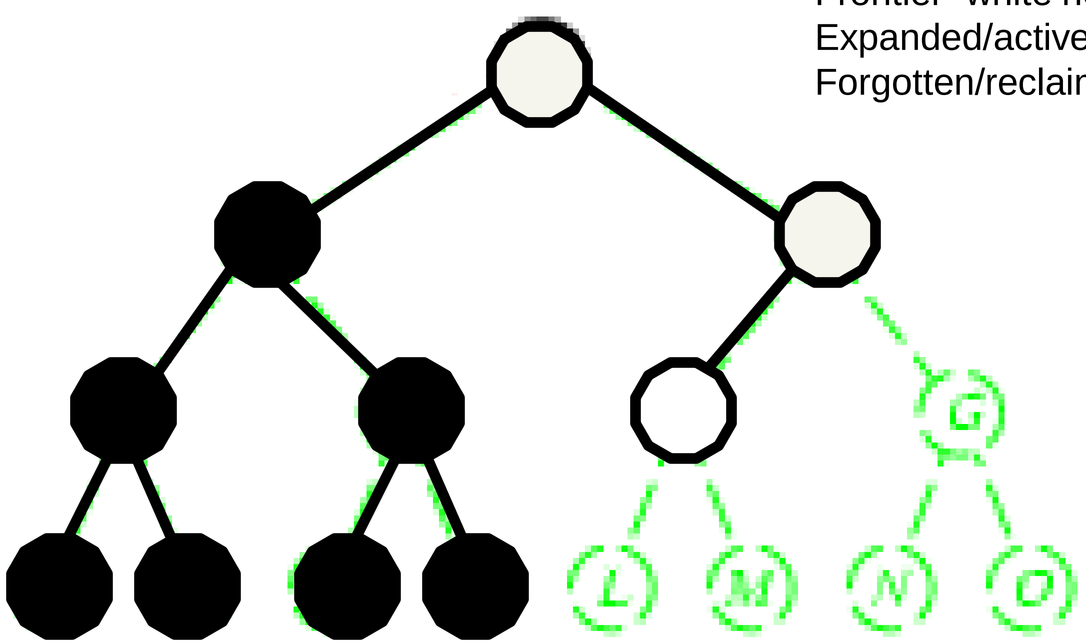

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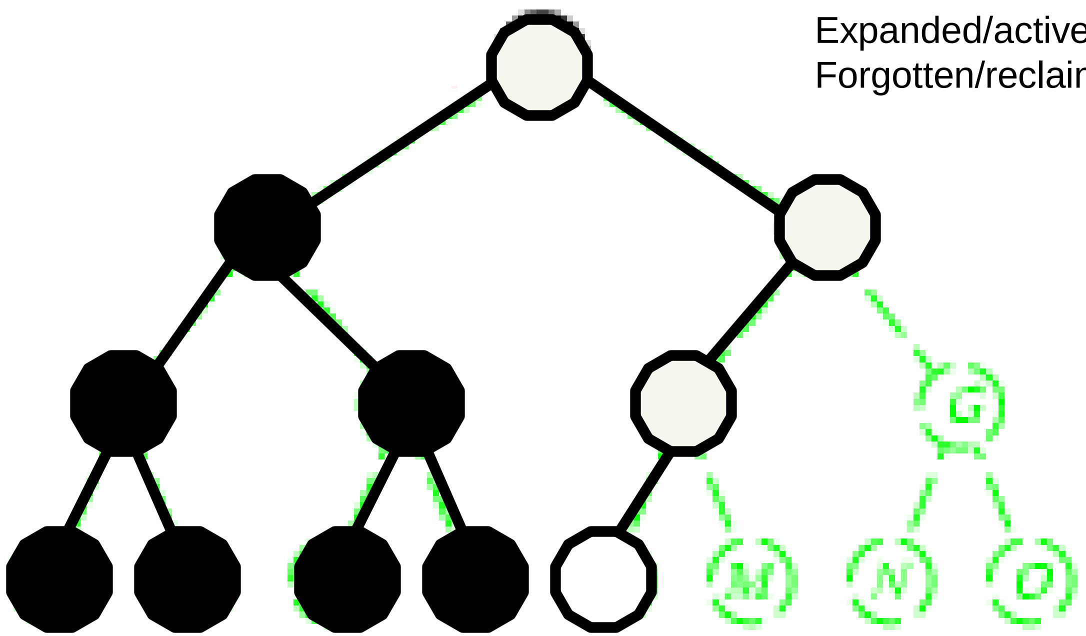

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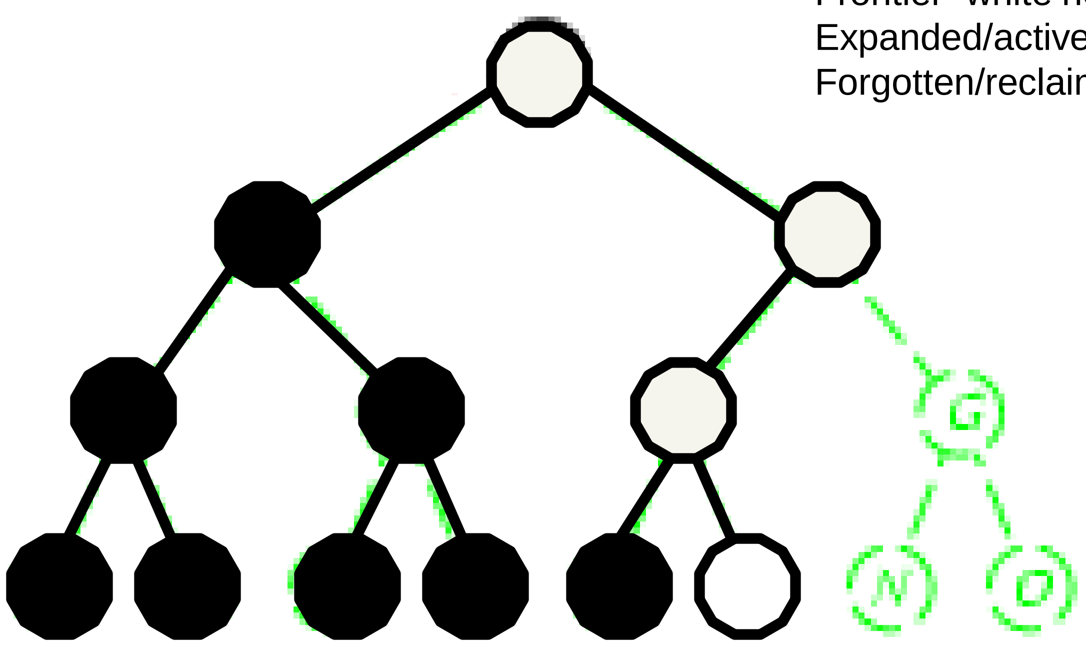

Improving Backtracking O(exp(n))

- Make our search more "informed" (e.g. heuristics)
  - General purpose methods can give large speed gains
  - CSPs are a generic formulation; hence heuristics are more "generic" as well

- Before search:
  - **Reduce the search space**
  - Arc-consistency, path-consistency, i-consistency
  - Variable ordering (fixed)

- During search:
  - **Look-ahead schemes**:
    - Detecting failure early; reduce the search space if possible
    - Which variable should be assigned next?
    - Which value should we explore first?
  - **Look-back schemes**:
    - Backjumping
    - Constraint recording
    - Dependency-directed backtracking

Look-ahead: Variable and value orderings

- Intuition:
  - Apply propagation at each node in the search tree (reduce future branching)
  - Choose a variable that will detect failures early (low branching factor)
  - Choose value least likely to yield a dead-end (find solution early if possible)

- Forward-checking
  - (check each unassigned variable separately)
- Maintaining arc-consistency (MAC)
  - (apply full arc-consistency)

Backtracking search (Figure 6.5)

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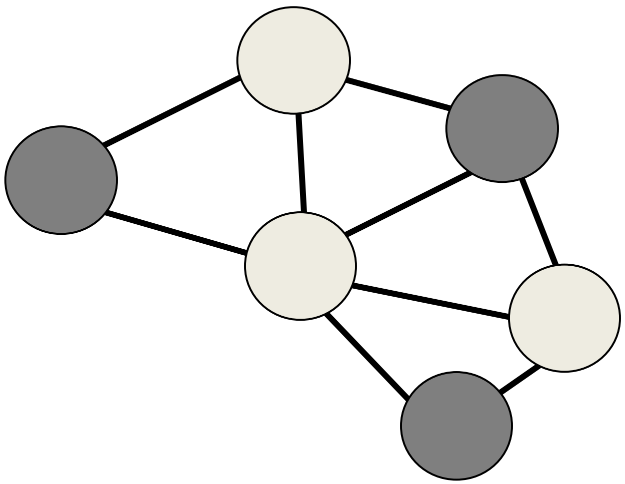
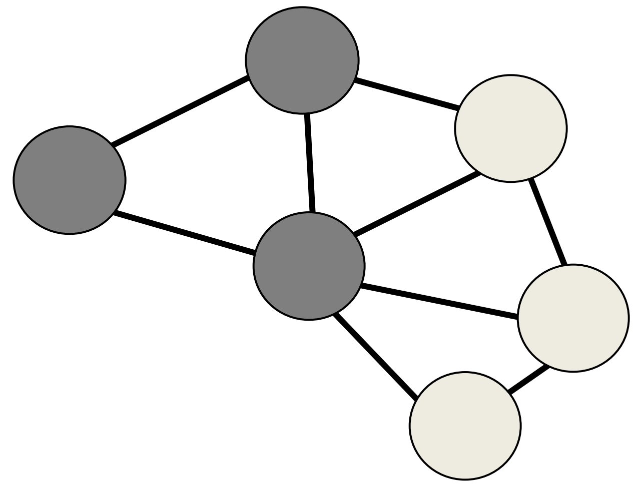
Dependence on variable ordering

- Example: coloring



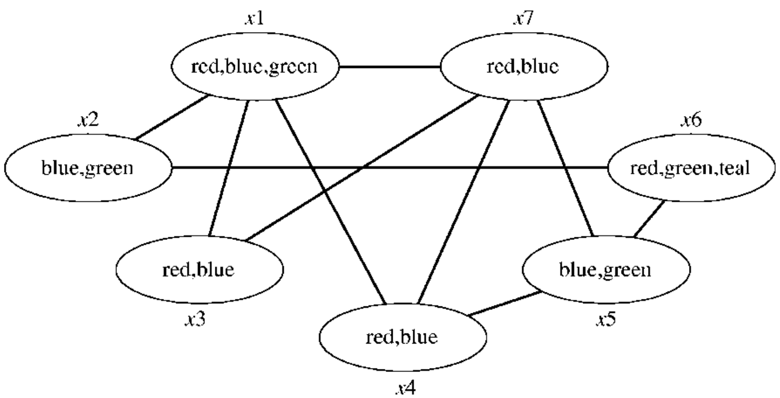
Color WA, Q, V first:
9 ways to color
none inconsistent (yet)
only 3 lead to solutions…

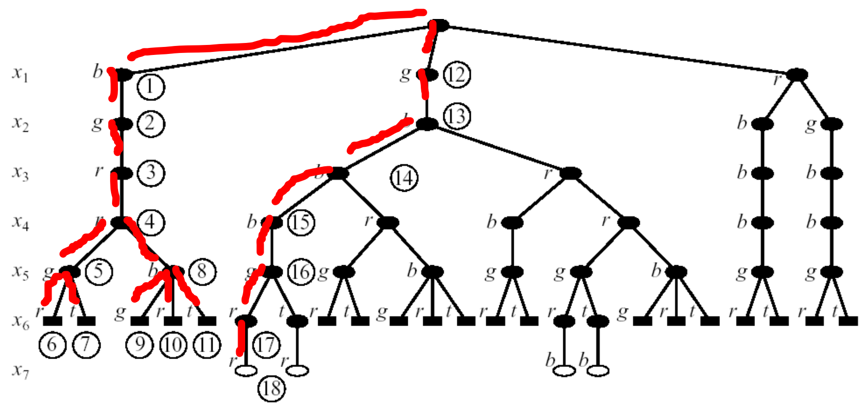


Color WA, SA, NT first:
6 ways to color
all lead to solutions
no backtracking

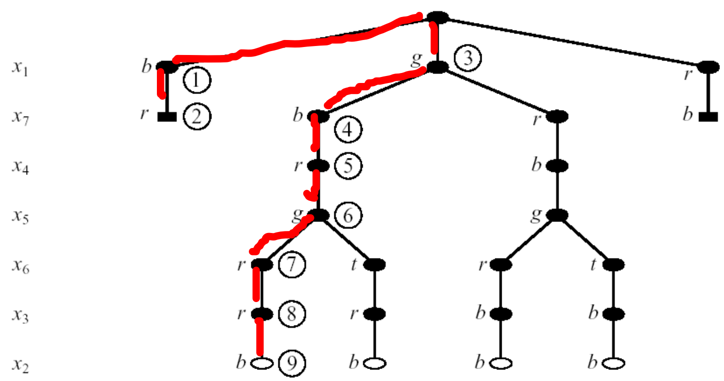
Dependence on variable ordering

- Another graph coloring example:





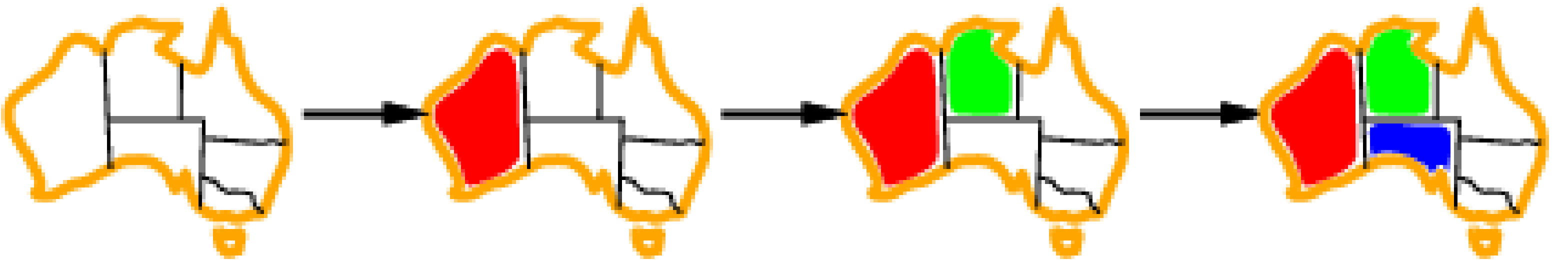
(a)



(b)

Minimum remaining values (MRV)

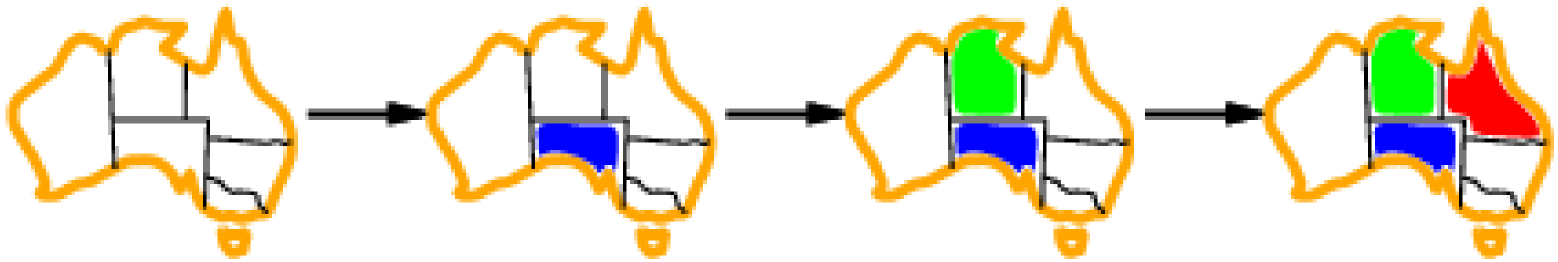
- A heuristic for selecting the next variable
  - a.k.a. most constrained variable (MCV) heuristic



  - choose the variable with the fewest legal values
  - will immediately detect failure if X has no legal values
  - (Related to forward checking, later)

Degree heuristic

- Another heuristic for selecting the next variable
  - a.k.a. most constraining variable heuristic



  - Select variable involved in the most constraints on other unassigned variables

  - **<u>Useful as a tie-breaker among most constrained variables</u>**

What about the order to try values?

Backtracking search (Figure 6.5)

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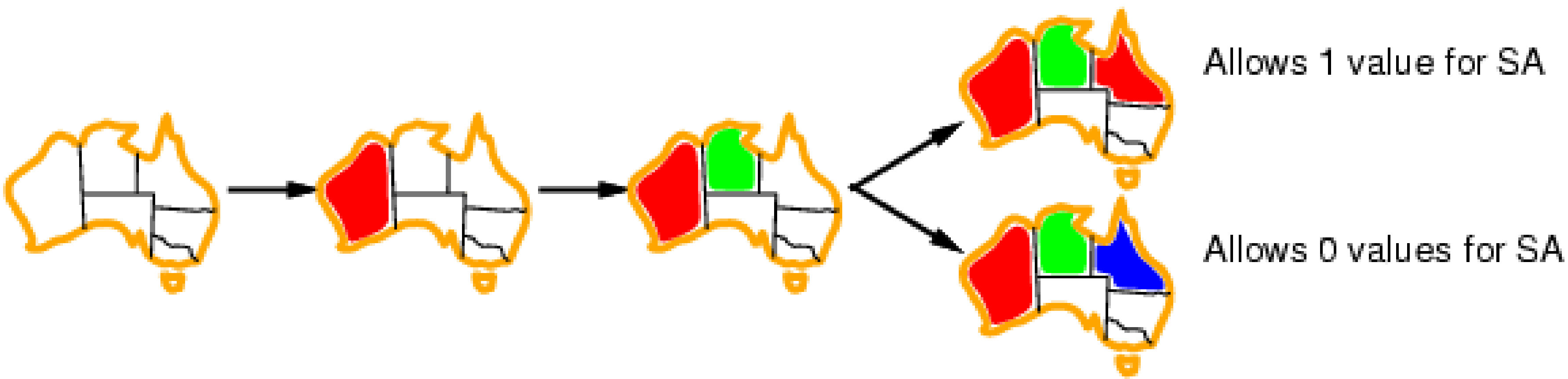
Least Constraining Value

- Heuristic for selecting what value to try next
- Given a variable, choose the least constraining value:
  – the one that rules out the fewest values in the remaining variables



  – Makes it more likely to find a solution early

Variable and value orderings

- Minimum remaining values for variable ordering
- Least constraining value for value ordering
  - Why do we want these? Is there a contradiction?

- Intuition:
  - Choose a variable that will detect failures early (low branching factor)
  - Choose value least likely to yield a dead-end (find solution early if possible)

- MRV for variable selection reduces current branching factor
  - Low branching factor throughout tree = fast search
  - Hopefully, when we get to variables with currently many values, forward checking or arc consistency will have reduced their domains & they'll have low branching too
- LCV for value selection increases the chance of success
  - If we're going to fail at this node, we'll have to examine every value anyway
  - If we're going to succeed, the earlier we do, the sooner we can stop searching

Summary

- CSPs
  - special kind of problem: states defined by values of a fixed set of variables, goal test defined by constraints on variable values

- Backtracking = depth-first search with one variable assigned per node

- Heuristics
  - Variable ordering and value selection heuristics help significantly

- Variable ordering (selection) heuristics
  - Choose variable with Minimum Remaining Values (MRV)
  - Degree Heuristic – break ties after applying MRV

- Value ordering (selection) heuristic
  - Choose Least Constraining Value