Propositional Logic:
Methods of Proof (Part II)

You will be expected to know

- Basic definitions
  - Inference, derive, sound, complete
- Conjunctive Normal Form (CNF)
  - Convert a Boolean formula to CNF
- Do a short resolution proof
- Horn Clauses
- Do a short forward-chaining proof
- Do a short backward-chaining proof

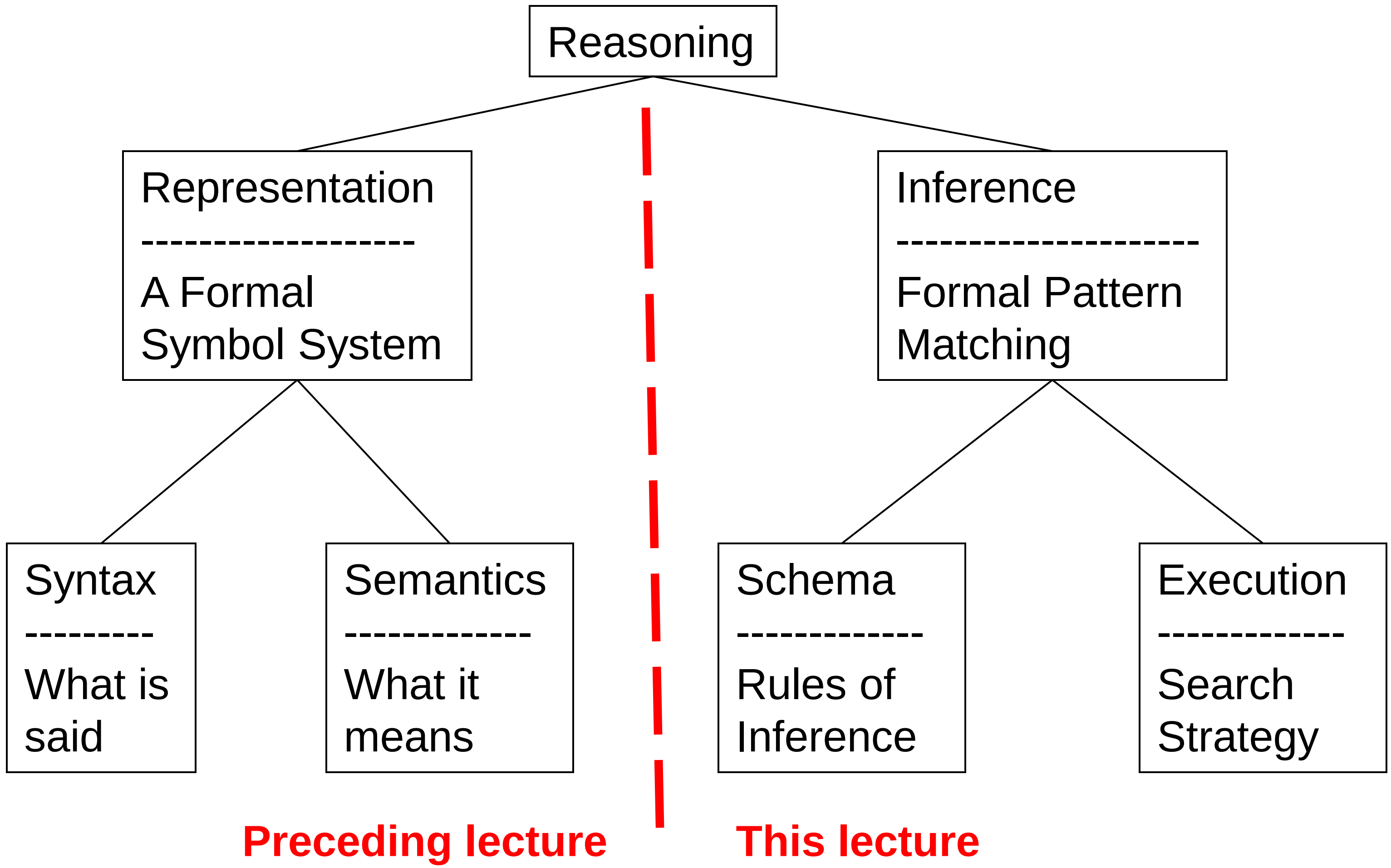
Review: Inference in Formal Symbol Systems Ontology, Representation, Inference

- **Formal Symbol Systems**
  - **Symbols** correspond to **things/ideas** in the world
  - **Pattern matching & rewrite** corresponds to **inference**

- **Ontology:** What exists in the world?
  - What must be represented?
- **Representation:** Syntax vs. Semantics
  - What’s Said vs. What’s Meant
- **Inference:** Schema vs. Mechanism
  - Proof Steps vs. Search Strategy

Ontology:
What kind of things exist in the world?
What do we need to describe and reason about?

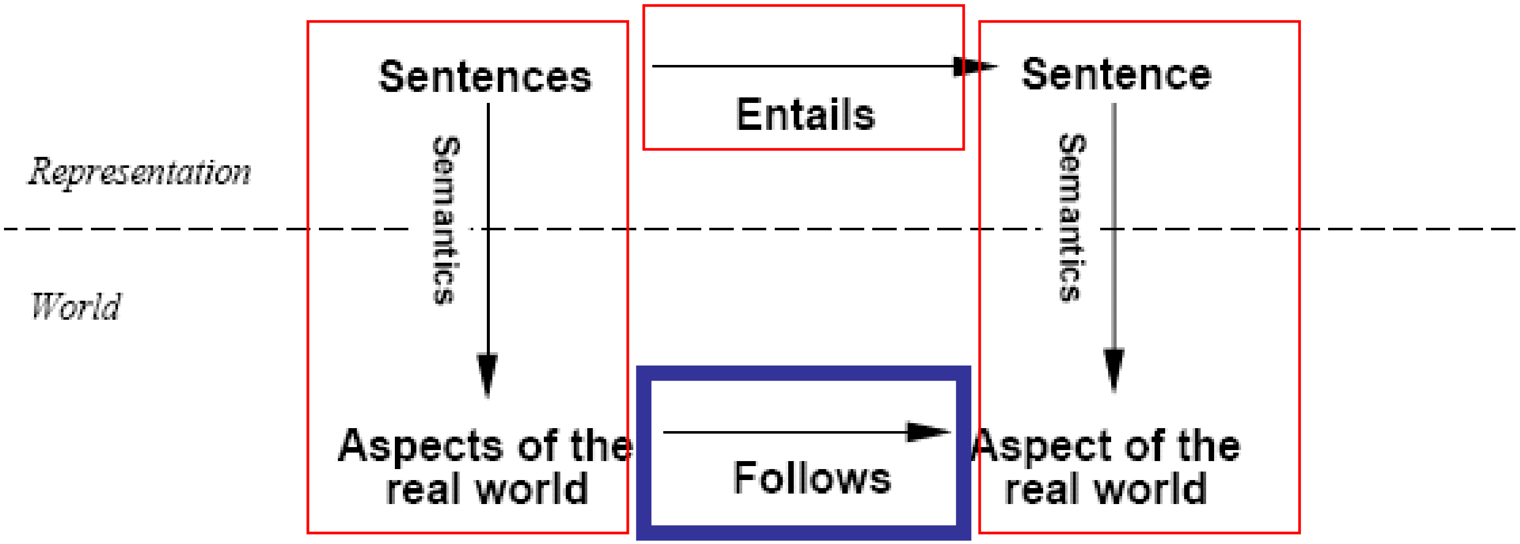
Review



Review

- Definitions:
  - Syntax, Semantics, Sentences, Propositions, Entails, Follows, Derives, Inference, Sound, Complete, Model, Satisfiable, Valid (or Tautology)
- Syntactic Transformations:
  - E.g., $(A \Rightarrow B) \Leftrightarrow (\neg A \vee B)$
- Semantic Transformations:
  - E.g., $(KB \models \alpha) \equiv (\models (KB \Rightarrow \alpha))$
- Truth Tables
  - Negation, Conjunction, Disjunction, Implication, Equivalence (Biconditional)
  - Inference by Model Enumeration

Review: Schematic perspective



If KB is true in the real world,

then any sentence α ***entailed*** *by KB*

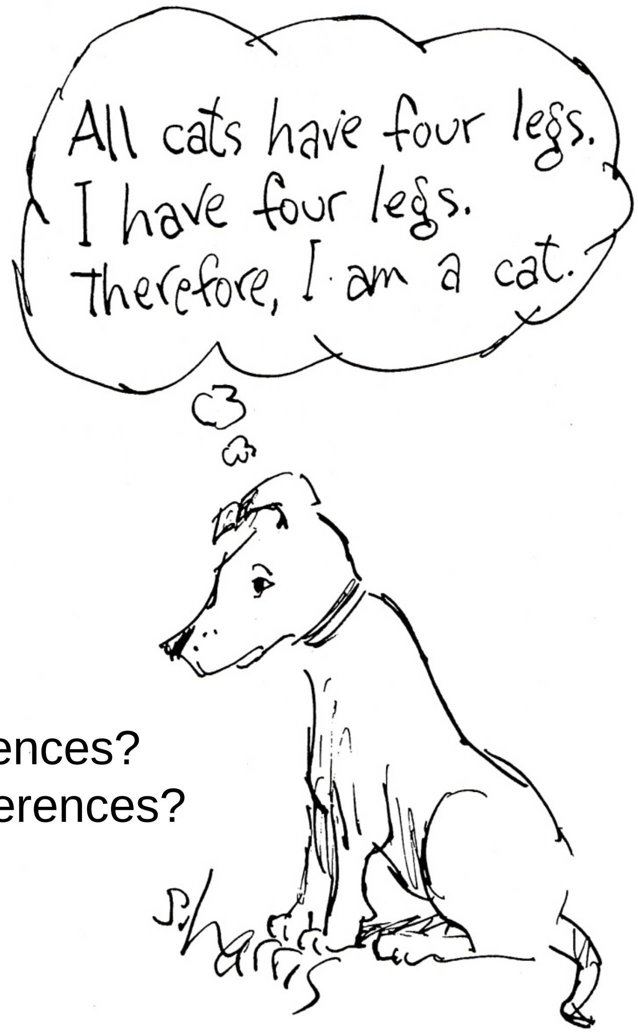
is also true in the real world.

For example: If I tell you (1) Sue is Mary's sister, and (2) Sue is Amy's mother, then it necessarily follows in the world that Mary is Amy's aunt, even though I told you nothing at all about aunts. This sort of reasoning pattern is what we hope to capture.

So --- how do we keep it from “Just making things up.” ?

Is this inference correct?

How do you know?
How can you tell?



How can we **make correct** inferences?
How can we **avoid incorrect** inferences?

“Einstein Simplified: Cartoons on Science” by Sydney Harris, 1992, Rutgers University Press

So --- how do we keep it from "Just making things up." ?

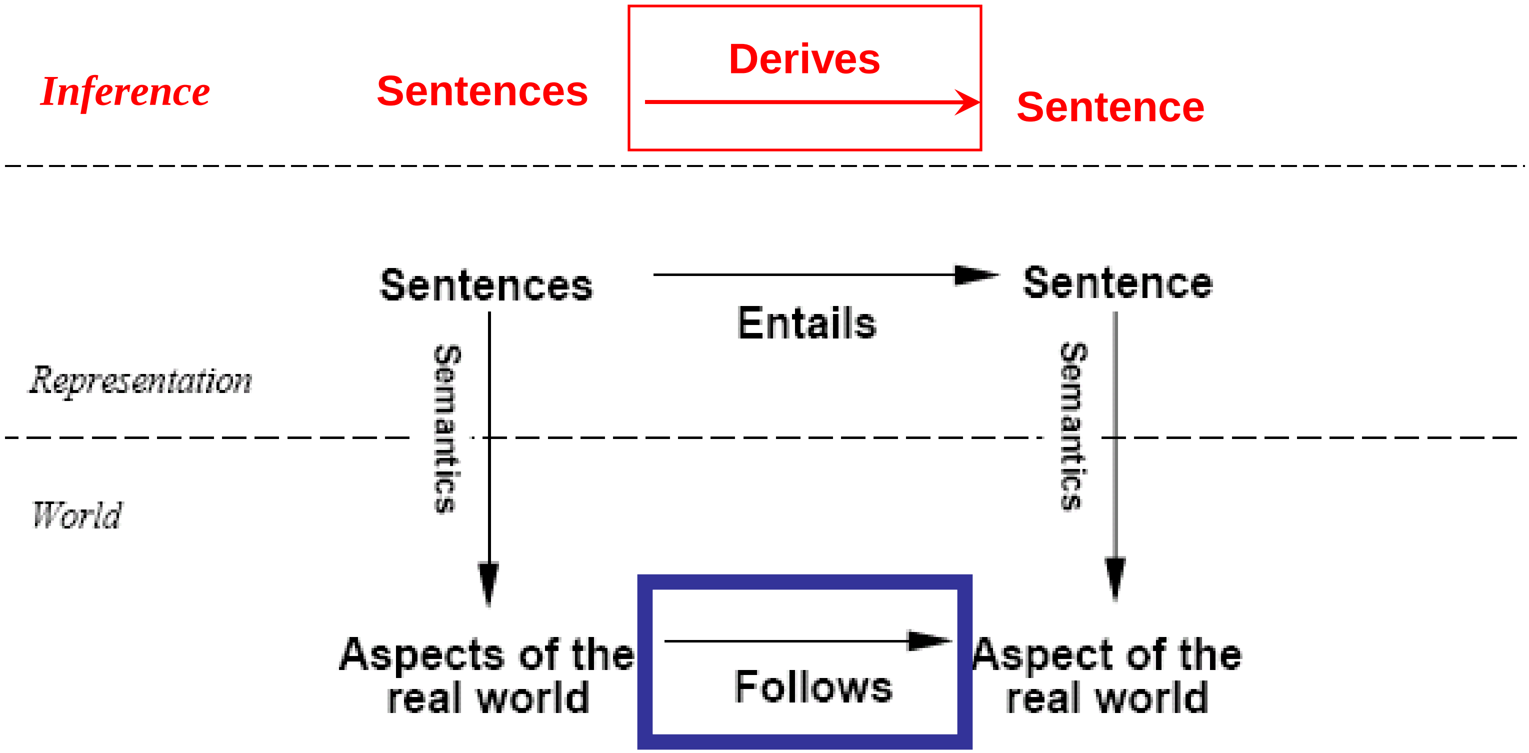
Is this inference correct?

How do you know?
How can you tell?

- All men are people;
  Half of all people are women;
  Therefore, half of all men are women.

- Penguins are black and white;
  Some old TV shows are black and white;
  Therefore, some penguins are old TV shows.

Schematic perspective



If KB is true in the real world,

then any sentence α ***derived*** *from KB*

by a sound inference procedure

is also true in the real world.

Logical inference

- The notion of entailment can be used for logic inference.
  - Model checking (see wumpus example):
    enumerate all possible models and check whether α is true.
- *KB* $\vdash_i \alpha$ means *KB* derives a sentence α using inference procedure *i*

- *Sound* (or *truth preserving*):
  The algorithm **only** derives entailed sentences.
  - Otherwise it just makes things up.
    i is sound iff whenever KB $\vdash_i \alpha$ *it is also true that KB* $\models \alpha$
  - *E.g., model-checking is sound*

  Refusing to infer any sentence is Sound; so, Sound is weak alone.
- *Complete*:
  The algorithm can derive **every** entailed sentence.
  i is complete iff whenever KB $\models \alpha$ *it is also true that KB* $\vdash_i \alpha$
  Deriving every sentence is Complete; so, Complete is weak alone.

Proof methods

- Proof methods divide into (roughly) two kinds:

Application of inference rules:

Legitimate (sound) generation of new sentences from old.

- Resolution --- KB is in Conjunctive Normal Form (CNF)
- Forward & Backward chaining

Model checking

Searching through truth assignments.

- Improved backtracking: Davis-Putnam-Logemann-Loveland (DPLL)
- Heuristic search in model space: Walksat.

Examples of Sound Inference Patterns

<u>Classical Syllogism (due to Aristotle)</u>

All Ps are Qs	**All Men are Mortal**
X is a P	**Socrates is a Man**
Therefore, X is a Q	**Therefore, Socrates is Mortal**

Why is this different from:
All men are people
Half of people are women
So half of men are women

<u>Implication (Modus Ponens)</u>

P implies Q	**Smoke implies Fire**
P	**Smoke**
Therefore, Q	**Therefore, Fire**

<u>Contrapositive (Modus Tollens)</u>

P implies Q	**Smoke implies Fire**
Not Q	**Not Fire**
Therefore, Not P	**Therefore, not Smoke**

<u>Law of the Excluded Middle (due to Aristotle)</u>

A Or B	**Alice is a Democrat or a Republican**
Not A	**Alice is not a Democrat**
Therefore, B	**Therefore, Alice is a Republican**

Inference by Resolution

- KB is represented in CNF
  - KB = AND of all the sentences in KB
  - KB sentence = clause = OR of literals
  - Literal = propositional symbol or its negation
- Find two clauses in KB, one of which contains a literal and the other its negation
- Cancel the literal and its negation
- Bundle everything else into a new clause
- Add the new clause to KB

Conjunctive Normal Form (CNF)

- Boolean formulae are central to CS
  - Boolean logic is the way our discipline works
- Two canonical Boolean formulae representations:
  - CNF = Conjunctive Normal Form
    Clause
    - A conjunct of disjuncts = (AND (OR …) (OR …))
    - "…" = a list of literals (= a variable or its negation)
    - CNF is used by Resolution Theorem Proving
  - DNF = Disjunctive Normal Form
    Term
    - A disjunct of conjuncts = (OR (AND …) (AND …))
    - DNF is used by Decision Trees in Machine Learning
- Can convert any Boolean formula to CNF or DNF

Conjunctive Normal Form (CNF)

We'd like to prove: $KB \models \alpha$
(This is equivalent to $KB \land \neg\alpha$ is unsatisfiable.)

We first rewrite $KB \land \neg\alpha$ into **conjunctive normal form (CNF).**

literals

A "conjunction of disjunctions"

$$\underbrace{(A \lor \neg B)}_{\text{Clause}} \land \underbrace{(B \lor \neg C \lor \neg D)}_{\text{Clause}}$$

- Any KB can be converted into CNF.
- In fact, any KB can be converted into CNF-3 using clauses with at most 3 literals.

Example: Conversion to CNF

Example: $B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$

1. Eliminate $\Leftrightarrow$ by replacing $\alpha \Leftrightarrow \beta$ with $(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)$.

   $= (B_{1,1} \Rightarrow (P_{1,2} \vee P_{2,1})) \wedge ((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1})$

2. Eliminate $\Rightarrow$ by replacing $\alpha \Rightarrow \beta$ with $\neg\alpha \vee \beta$ and simplify.

   $= (\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}) \wedge (\neg(P_{1,2} \vee P_{2,1}) \vee B_{1,1})$

3. Move $\neg$ inwards using de Morgan's rules and simplify.

   $$\neg(\alpha \vee \beta) = \neg\alpha \wedge \neg\beta$$

   $= (\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}) \wedge ((\neg P_{1,2} \wedge \neg P_{2,1}) \vee B_{1,1})$

4. Apply distributive law ($\wedge$ over $\vee$) and simplify.

   $= (\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}) \wedge (\neg P_{1,2} \vee B_{1,1}) \wedge (\neg P_{2,1} \vee B_{1,1})$

Example: Conversion to CNF

Example: $B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$

From the previous slide we had:

$= (\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}) \wedge (\neg P_{1,2} \vee B_{1,1}) \wedge (\neg P_{2,1} \vee B_{1,1})$

5. KB is the conjunction of all of its sentences (all are true), so write each clause (disjunct) as a sentence in KB:

KB =

...

$(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1})$ $\longrightarrow$ $(\neg B_{1,1} \quad P_{1,2} \quad P_{2,1})$

$(\neg P_{1,2} \vee B_{1,1})$ $\longrightarrow$ $(\neg P_{1,2} \quad B_{1,1})$

$(\neg P_{2,1} \vee B_{1,1})$ $\xrightarrow[\text{(same)}]{}$ $(\neg P_{2,1} \quad B_{1,1})$

...

Often, Won't Write "$\vee$" or "$\wedge$"

(we know they are there)

Inference by Resolution

- KB is represented in CNF
  - KB = AND of all the sentences in KB
  - KB sentence = clause = OR of literals
  - Literal = propositional symbol or its negation
- Find two clauses in KB, one of which contains a literal and the other its negation
- Cancel the literal and its negation
- Bundle everything else into a new clause
- Add the new clause to KB

Resolution = Efficient Implication

Recall that (A => B) = ((NOT A) OR B)
and so:

(Y OR X) = ((NOT X) => Y)

((NOT Y) OR Z) = (Y => Z)

which yields:

((Y OR X) AND ((NOT Y) OR Z)) = ((NOT X) => Z) = (X OR Z)

(OR A B C D)	->Same ->	(NOT (OR B C D)) => A
(OR ¬A E F G)	->Same ->	A => (OR E F G)
----------------------------		--
(OR B C D E F G)		(NOT (OR B C D)) => (OR E F G)
		--
		(OR B C D E F G)

Recall: All clauses in KB are conjoined by an implicit AND (= CNF representation).

Resolution Examples

- Resolution: inference rule for CNF: sound and complete! *

$(A \vee B \vee C)$

$(\neg A)$

“If A or B or C is true, but not A, then B or C must be true.”

$$\frac{(A \vee B \vee C) \quad (\neg A)}{\therefore (B \vee C)}$$

$(A \vee B \vee C)$

$(\neg A \vee D \vee E)$

“If A is false then B or C must be true, or if A is true then D or E must be true, hence since A is either true or false, B or C or D or E must be true.”

$\therefore (B \vee C \vee D \vee E)$

$(A \vee B)$

$(\neg A \vee B)$

“If A or B is true, and not A or B is true, then B must be true.”

$\therefore (B \vee B) \equiv B$ ← Simplification is done always.

* Resolution is “refutation complete” in that it can prove the truth of any entailed sentence by refutation.
* You can start two resolution proofs in parallel, one for the sentence and one for its negation, and see which branch returns a correct proof.

Only Resolve ONE Literal Pair!

If more than one pair, result always = TRUE.

Useless!! Always simplifies to TRUE!!

No!

(OR A B C D)
(OR ¬A ¬B F G)

(OR C D F G)

No!

No!

(OR A B C D)
(OR ¬A ¬B ¬C)

(OR D)

No!

Yes! (but = TRUE)

(OR A B C D)
(OR ¬A ¬B F G)

(OR B ¬B C D F G)

Yes! (but = TRUE)

Yes! (but = TRUE)

(OR A B C D)
(OR ¬A ¬B ¬C)

(OR A ¬A B ¬B D)

Yes! (but = TRUE)

Resolution Algorithm

- The resolution algorithm tries to prove: $KB \models \alpha$ *equivalent to* $KB \wedge \neg\alpha$ *unsatisfiable*

- Generate all new sentences from KB and the (negated) query.
- One of two things can happen:

1. We find $P \wedge \neg P$ which is unsatisfiable. I.e.* we **can** entail the query.

2. We find no contradiction: there is a model that satisfies the sentence $KB \wedge \neg\alpha$ (non-trivial) and hence we **cannot** entail the query.

* I.e. = *id est* = that is

Resolution example

Stated in English

- "Laws of Physics" in the Wumpus World:
  - "A breeze in B11 is equivalent to a pit in P12 or a pit in P21."
- Particular facts about a specific instance:
  - "There is no breeze in B11."
- Goal or query sentence:
  - "Is it true that P12 does not have a pit?"

Resolution example

Stated in Propositional Logic

- "Laws of Physics" in the Wumpus World:
  - "A breeze in B11 is equivalent to a pit in P12 or a pit in P21."

    $(B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1}))$

We converted this sentence to CNF in the CNF example we worked above.

- Particular facts about a specific instance:
  - "There is no breeze in B11."

    $(\neg B_{1,1})$
- Goal or query sentence:
  - "Is it true that P12 does not have a pit?"

    $(\neg P_{1,2})$

Resolution example

Resulting Knowledge Base stated in CNF

- "Laws of Physics" in the Wumpus World:

  $(\neg B_{1,1} \quad P_{1,2} \quad P_{2,1})$
  $(\neg P_{1,2} \quad B_{1,1})$
  $(\neg P_{2,1} \quad B_{1,1})$

- Particular facts about a specific instance:

  $(\neg\ B_{1,1})$

- Negated goal or query sentence:

  $(P_{1,2})$

Resolution example

A Resolution proof ending in ( )

- Knowledge Base at start of proof:

$(\neg B_{1,1} \quad P_{1,2} \quad P_{2,1})$

$(\neg P_{1,2} \quad B_{1,1})$

$(\neg P_{2,1} \quad B_{1,1})$

$(\neg\ B_{1,1})$

$(P_{1,2})$

A resolution proof ending in ( ):

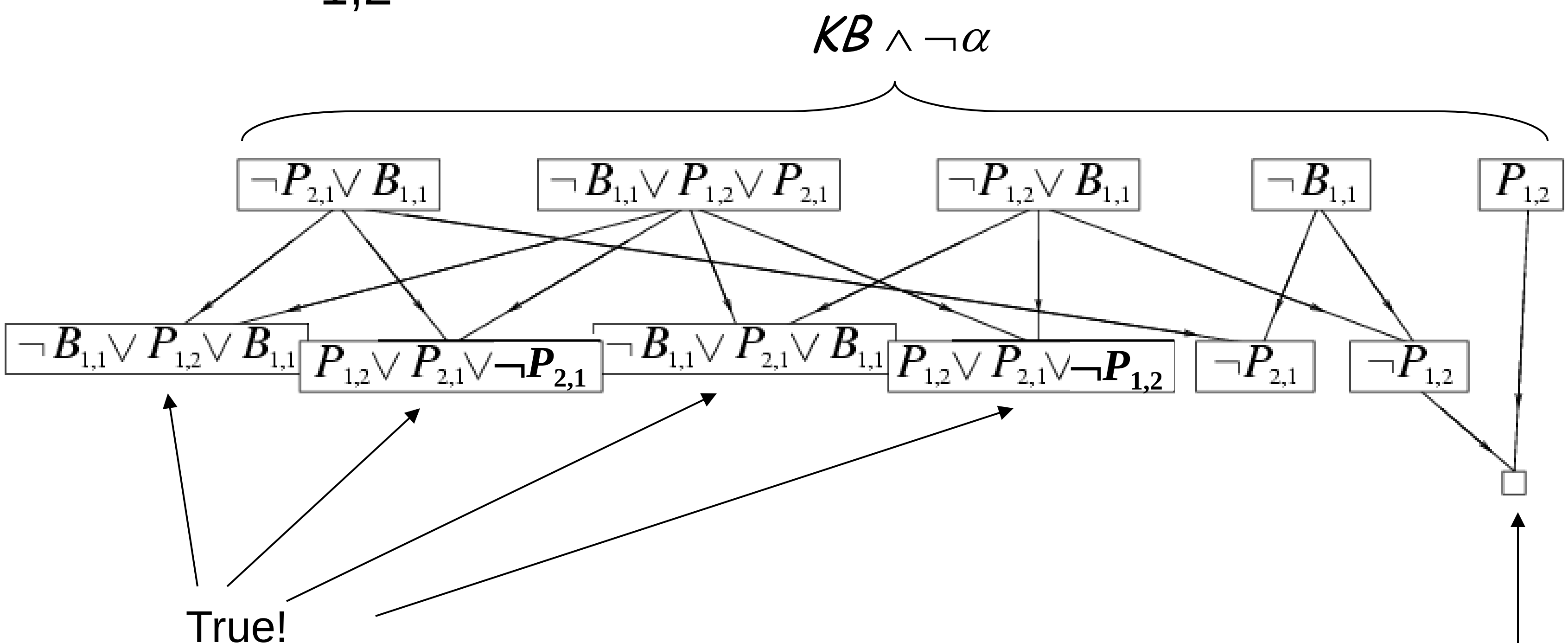
- Resolve $(\neg P_{1,2} \quad B_{1,1})$ and $(\neg\ B_{1,1})$ to give $(\neg P_{1,2}\ )$
- Resolve $(\neg P_{1,2}\ )$ and $(P_{1,2})$ to give $(\ )$

- Consequently, the goal or query sentence is entailed by KB.
- Of course, there are many other proofs, which are OK iff correct.

Resolution example

Graphical view of the proof

- $KB = (B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})) \wedge \neg B_{1,1}$
- $\alpha = \neg P_{1,2}$



A sentence in KB is not "used up" when it is used in a resolution step. It is true, remains true, and is still in KB.

False in all worlds

Detailed Resolution Proof Example

- **In words:** *If the unicorn is mythical, then it is immortal, but if it is not mythical, then it is a mortal mammal. If the unicorn is either immortal or a mammal, then it is horned. The unicorn is magical if it is horned.*

  <u>*Prove that the unicorn is both magical and horned.*</u>

Problem 7.2, R&N page 280. (Adapted from Barwise and Etchemendy, 1993.)

Note for non-native-English speakers: immortal = not mortal

Detailed Resolution Proof Example

- **In words:** *If the unicorn is mythical, then it is immortal, but if it is not mythical, then it is a mortal mammal. If the unicorn is either immortal or a mammal, then it is horned. The unicorn is magical if it is horned.*

  Prove that the unicorn is both magical and horned.

- **First, Ontology**: What do we need to describe and reason about?

- Use these propositional variables ("immortal" = "not mortal"):

  Y = unicorn is mYthical   R = unicorn is moRtal

  M = unicorn is a maMmal   H = unicorn is Horned

  G = unicorn is maGical

Detailed Resolution Proof Example

- **In words:** *If the unicorn is mythical, then it is immortal, but if it is not mythical, then it is a mortal mammal. If the unicorn is either immortal or a mammal, then it is horned. The unicorn is magical if it is horned.*

  Prove that the unicorn is both magical and horned.

  Y = unicorn is mYthical   R = unicorn is moRtal

  M = unicorn is a maMmal   H = unicorn is Horned

  G = unicorn is maGical

- **Second, translate to Propositional Logic, then to CNF:**
- Propositional logic (prefix form, aka Polish notation):
  - **(=> Y (NOT R))** ; same as (Y => (NOT R)) in infix form
- CNF (clausal form) ; recall (A => B) = ((NOT A) OR B)
  - **((NOT Y) (NOT R))**

Prefix form is often a better representation for a parser, since it looks at the first element of the list and dispatches to a handler for that operator token.

Detailed Resolution Proof Example

- **In words:** *If the unicorn is mythical, then it is immortal, but if it is not mythical, then it is a mortal mammal. If the unicorn is either immortal or a mammal, then it is horned. The unicorn is magical if it is horned.*

  Prove that the unicorn is both magical and horned.

  Y = unicorn is mYthical   R = unicorn is moRtal

  M = unicorn is a maMmal   H = unicorn is Horned

  G = unicorn is maGical

- **Second, translate to Propositional Logic, then to CNF**:
- Propositional logic (prefix form):
  – **(=> (NOT Y) (AND R M))**   ;same as ((NOT Y) => (R AND M)) in infix form
- CNF (clausal form)
  – **(M Y)**
  – **(R Y)**

If you ever have to do this "for real" you will likely invent a new domain language that allows you to state important properties of the domain --- then parse that into propositional logic, and then CNF.

Detailed Resolution Proof Example

- **In words:** *If the unicorn is mythical, then it is immortal, but if it is not mythical, then it is a mortal mammal. If the unicorn is either immortal or a mammal, then it is horned. The unicorn is magical if it is horned.*

  Prove that the unicorn is both magical and horned.

  Y = unicorn is mYthical

  R = unicorn is moRtal

  M = unicorn is a maMmal

  H = unicorn is Horned

  G = unicorn is maGical

- **Second, translate to Propositional Logic, then to CNF**:
- Propositional logic (prefix form):
  - **(=> (OR (NOT R) M) H)** ; same as ((Not R) OR M) => H in infix form
- CNF (clausal form)
  - **(H (NOT M))**
  - **(H R)**

Detailed Resolution Proof Example

- **In words:** *If the unicorn is mythical, then it is immortal, but if it is not mythical, then it is a mortal mammal. If the unicorn is either immortal or a mammal, then it is horned. The unicorn is magical if it is horned.*

  Prove that the unicorn is both magical and horned.

  Y = unicorn is mYthical R = unicorn is moRtal

  M = unicorn is a maMmal H = unicorn is Horned

  G = unicorn is maGical

- **Second, translate to Propositional Logic, then to CNF**:
- Propositional logic (prefix form)
  - **(=> H G)** ; same as H => G in infix form
- CNF (clausal form)
  - **((NOT H) G)**

Detailed Resolution Proof Example

- **In words:** *If the unicorn is mythical, then it is immortal, but if it is not mythical, then it is a mortal mammal. If the unicorn is either immortal or a mammal, then it is horned. The unicorn is magical if it is horned.*

  Prove that the unicorn is both magical and horned.

  Y = unicorn is mYthical　　R = unicorn is moRtal

  M = unicorn is a maMmal　　H = unicorn is Horned

  G = unicorn is maGical

- **Current KB** (in CNF clausal form) =

  ((NOT Y) (NOT R))　**(M Y)**　**(R Y)**　**(H (NOT M))**

  (H R)　**((NOT H) G)**

Detailed Resolution Proof Example

- **In words:** *If the unicorn is mythical, then it is immortal, but if it is not mythical, then it is a mortal mammal. If the unicorn is either immortal or a mammal, then it is horned. The unicorn is magical if it is horned.*

  Prove that the unicorn is both magical and horned.

  Y = unicorn is mYthical R = unicorn is moRtal

  M = unicorn is a maMmal H = unicorn is Horned

  G = unicorn is maGical

- **Third, negated goal to Propositional Logic, then to CNF:**
- Goal sentence in propositional logic (prefix form)
  – **(AND H G)** ; same as H AND G in infix form
- Negated goal sentence in propositional logic (prefix form)
  – **(NOT (AND H G)) = (OR (NOT H) (NOT G))**
- CNF (clausal form)
  – **((NOT G) (NOT H))**

Detailed Resolution Proof Example

- **In words:** *If the unicorn is mythical, then it is immortal, but if it is not mythical, then it is a mortal mammal. If the unicorn is either immortal or a mammal, then it is horned. The unicorn is magical if it is horned.*

  Prove that the unicorn is both magical and horned.

  Y = unicorn is mYthical R = unicorn is moRtal

  M = unicorn is a maMmal H = unicorn is Horned

  G = unicorn is maGical

- **Current KB + negated goal** (in CNF clausal form) =

((NOT Y) (NOT R))	**(M Y)**	**(R Y)**	**(H (NOT M))**
(H R)	**((NOT H) G)**	**((NOT G) (NOT H))**	

Detailed Resolution Proof Example

- **In words:** *If the unicorn is mythical, then it is immortal, but if it is not mythical, then it is a mortal mammal. If the unicorn is either immortal or a mammal, then it is horned. The unicorn is magical if it is horned.*

  <u>Prove that the unicorn is both magical and horned.</u>

((NOT Y) (NOT R))	**(M Y)**	**(R Y)**	**(H (NOT M))**
(H R)	**((NOT H) G)**	**((NOT G) (NOT H))**	

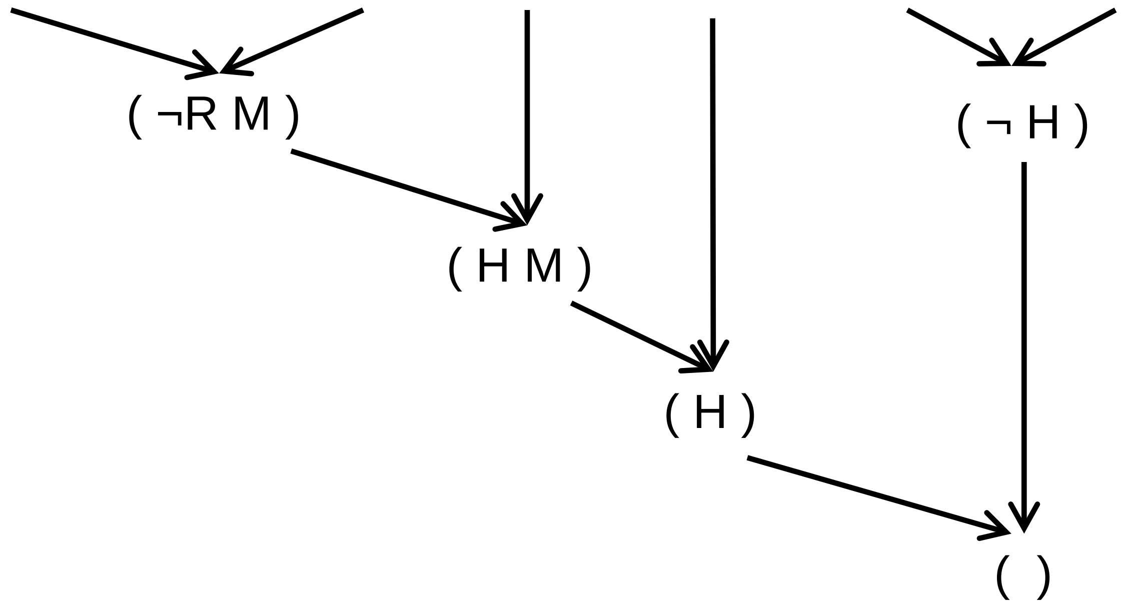
- **Fourth, produce a resolution proof ending in ( ):**
- Resolve (¬H ¬G) and (¬H G) to give (¬H)
- Resolve (¬Y ¬R) and (Y M) to give (¬R M)
- Resolve (¬R M) and (R H) to give (M H)
- Resolve (M H) and (¬M H) to give (H)
- Resolve (¬H) and (H) to give ( )

- Of course, there are many other proofs, which are OK iff correct.

Detailed Resolution Proof Example

Graph view of proof

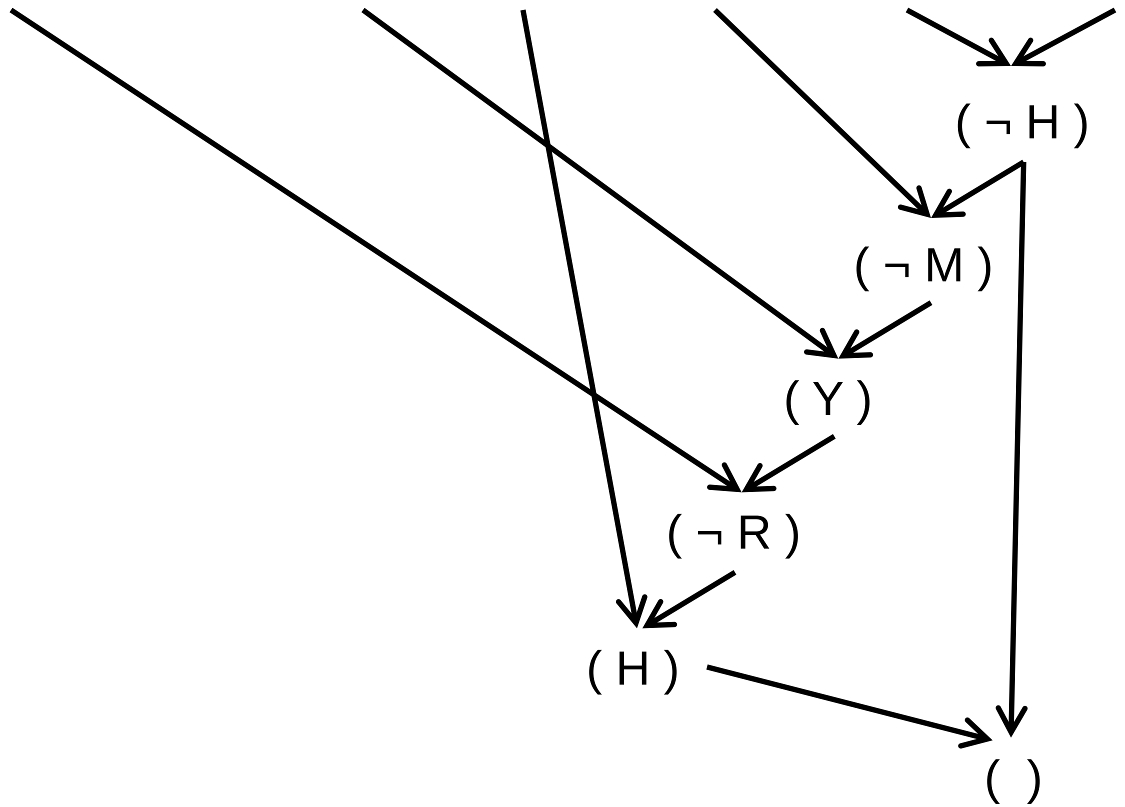
- (¬ Y ¬ R) (Y R) (Y M) (R H) (¬ M H) (¬ H G) (¬ G ¬ H)



Detailed Resolution Proof Example

Graph view of a different proof

- (¬ Y ¬ R) (Y R) (Y M) (R H) (¬ M H) (¬ H G) (¬ G ¬ H)



Horn Clauses

- Resolution can be exponential in space and time.

- If we can reduce all clauses to "Horn clauses" inference is linear in space and time

A clause with at most 1 positive literal.

e.g. $A \vee \neg B \vee \neg C$

- Every Horn clause can be rewritten as an implication with a conjunction of positive literals in the premises and at most a single positive literal as a conclusion.

  e.g. $A \vee \neg B \vee \neg C \equiv B \wedge C \Rightarrow A$
- 1 positive literal and $\geq$ 1 negative literal: definite clause (e.g., above)
- 0 positive literals: integrity constraint or goal clause

  e.g. $(\neg A \vee \neg B) \equiv (A \wedge B \Rightarrow False)$ states that ($A \wedge B$) must be false
- 0 negative literals: fact

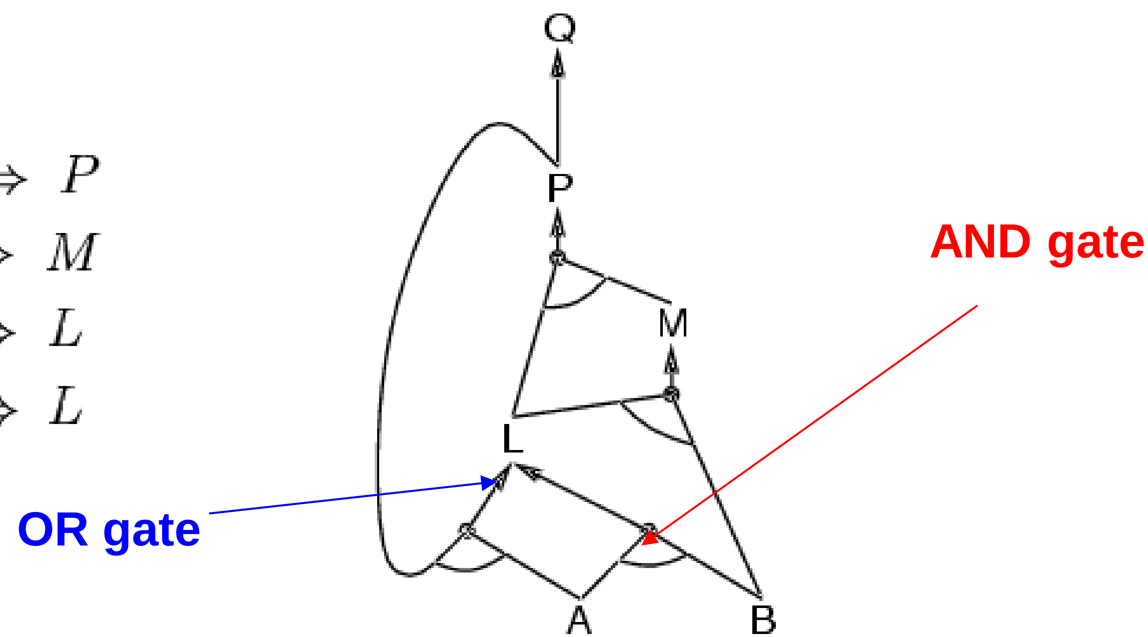
  e.g., (A) $\equiv$ (True $\Rightarrow$ A) states that A must be true.
- Forward Chaining and Backward chaining are sound and complete with Horn clauses and run linear in space and time.

Forward chaining (FC)

- Idea: fire any rule whose premises are satisfied in the *KB*, add its conclusion to the *KB*, until query is found.

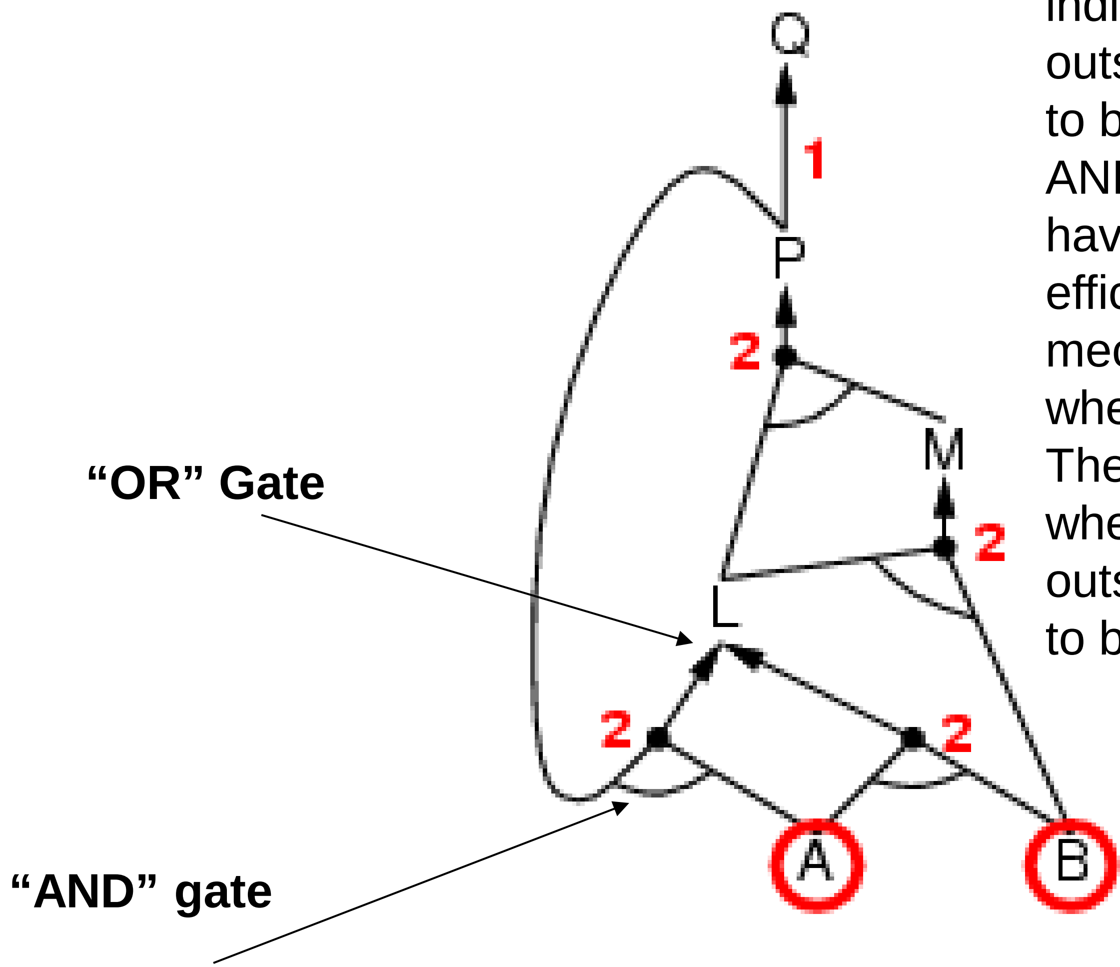
- This proves that $KB \Rightarrow Q$ is true in all possible worlds (i.e. trivial), and hence it proves entailment.

$$
\begin{aligned}
P &\Rightarrow Q \\
L \wedge M &\Rightarrow P \\
B \wedge L &\Rightarrow M \\
A \wedge P &\Rightarrow L \\
A \wedge B &\Rightarrow L \\
A & \\
B &
\end{aligned}
$$



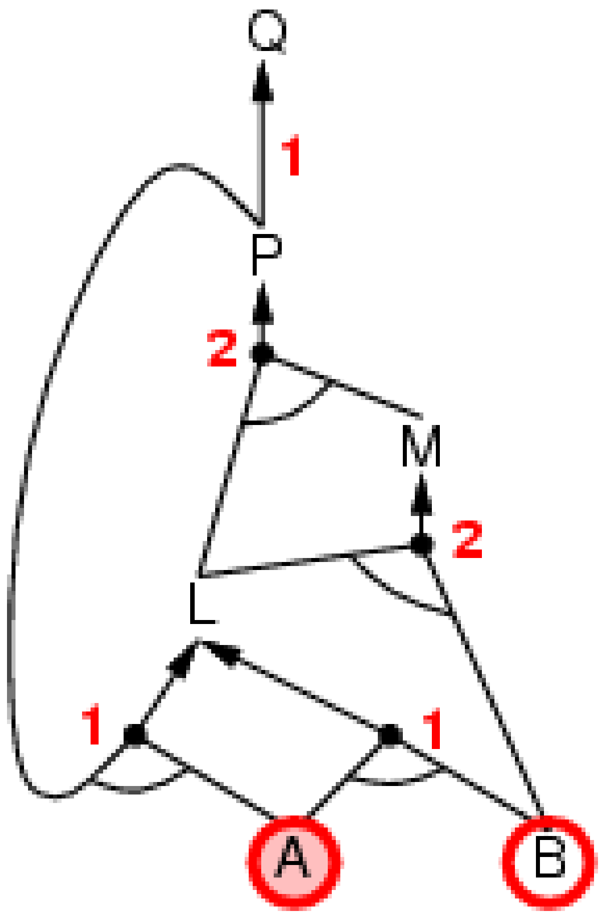
- Forward chaining is sound and complete for Horn KB

Forward chaining example

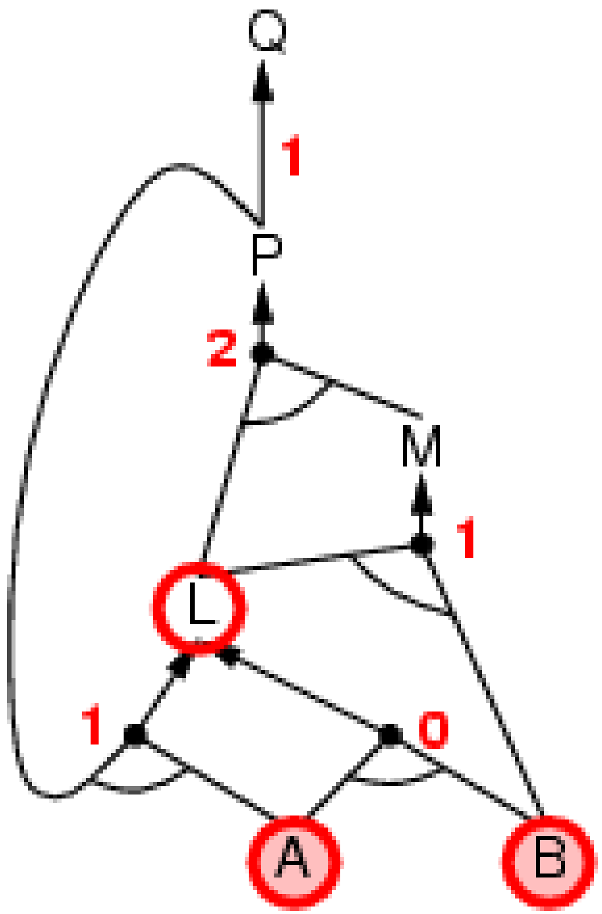


Numbers at each AND node indicate the number of outstanding preconditions yet to be satisfied before **all** of that AND node input preconditions have been satisfied. It is an efficient book-keeping mechanism for determining when an AND node is satisfied. The AND node is satisfied when its number of outstanding preconditions yet to be satisfied is zero.

Forward chaining example



Forward chaining example



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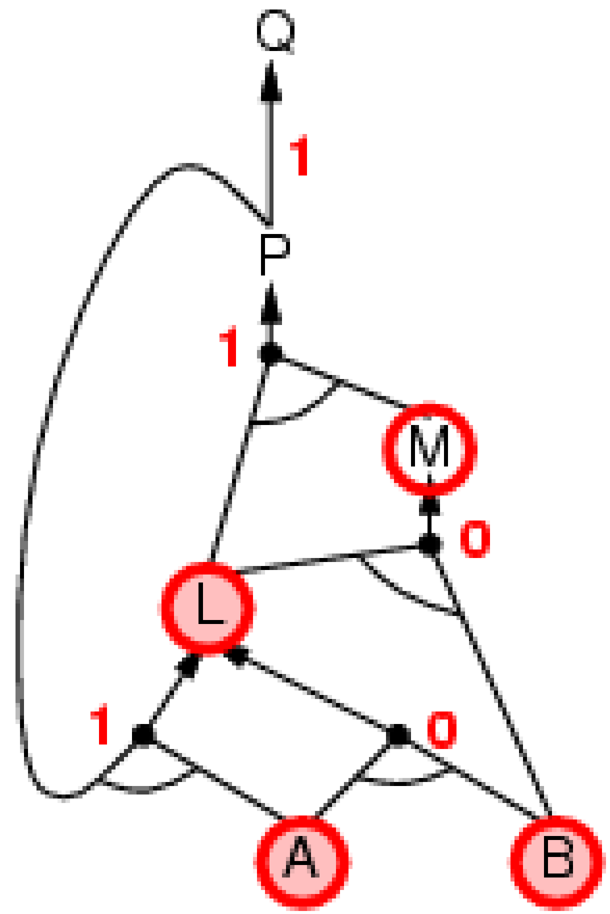
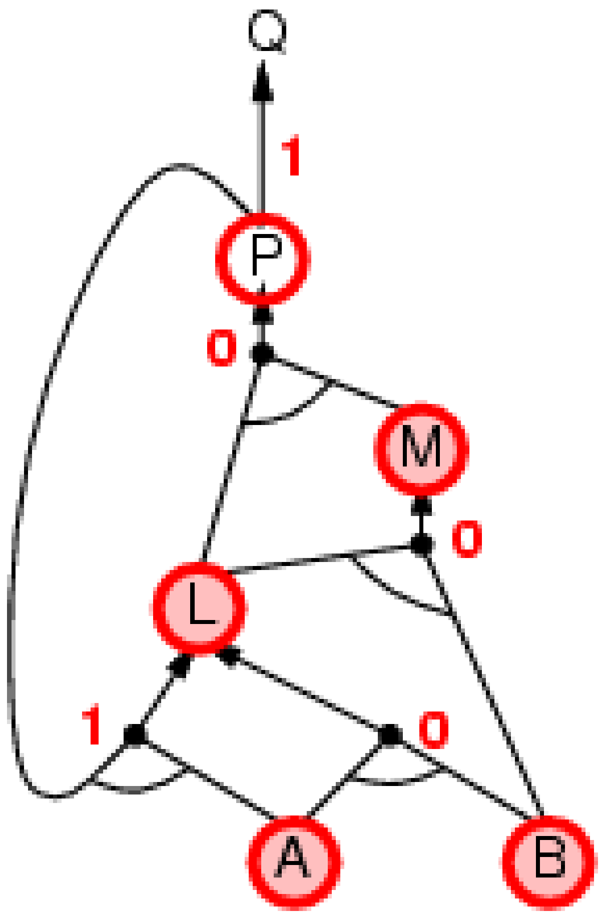
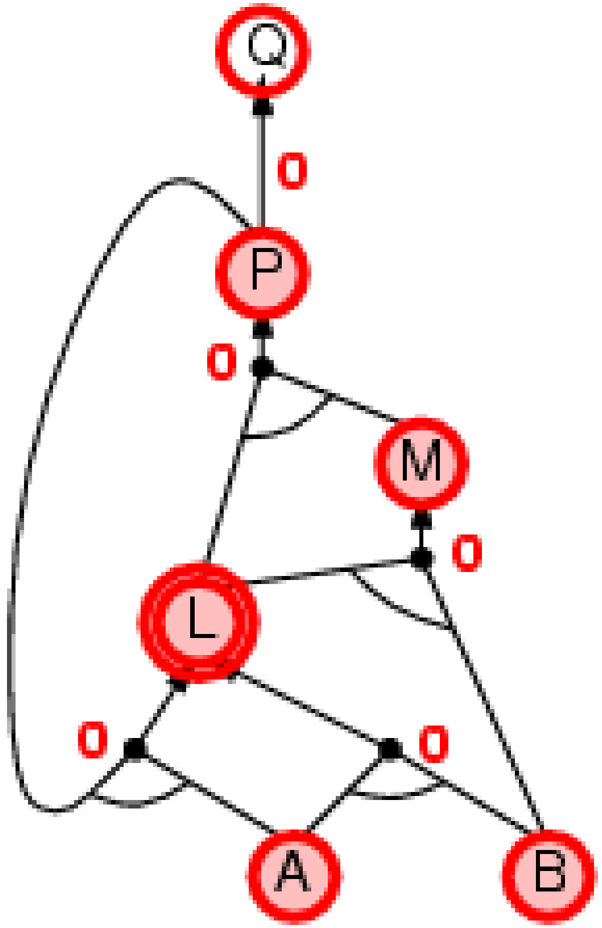


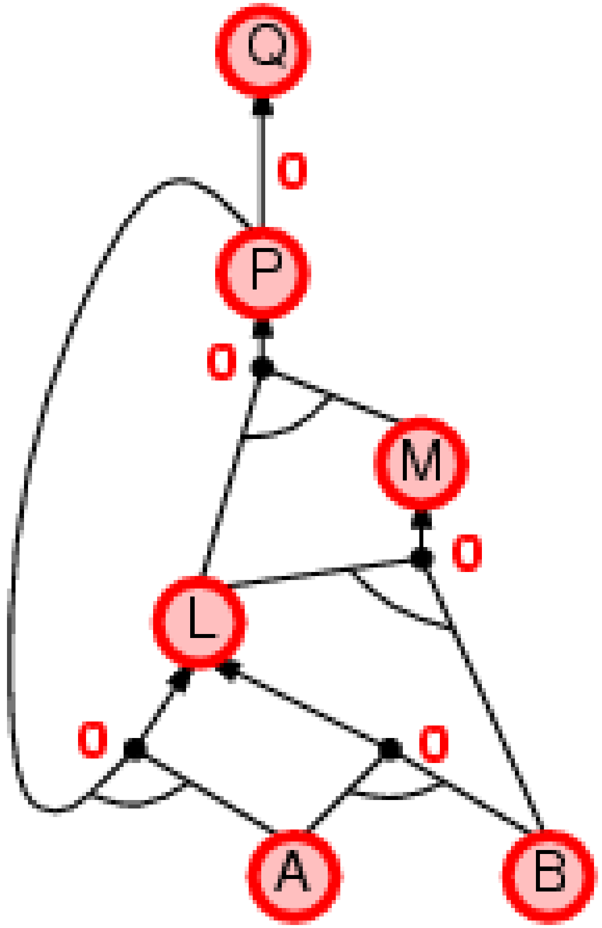
Forward chaining example



Forward chaining example



Forward chaining example



Backward chaining (BC)

Idea: work backwards from the query *q*

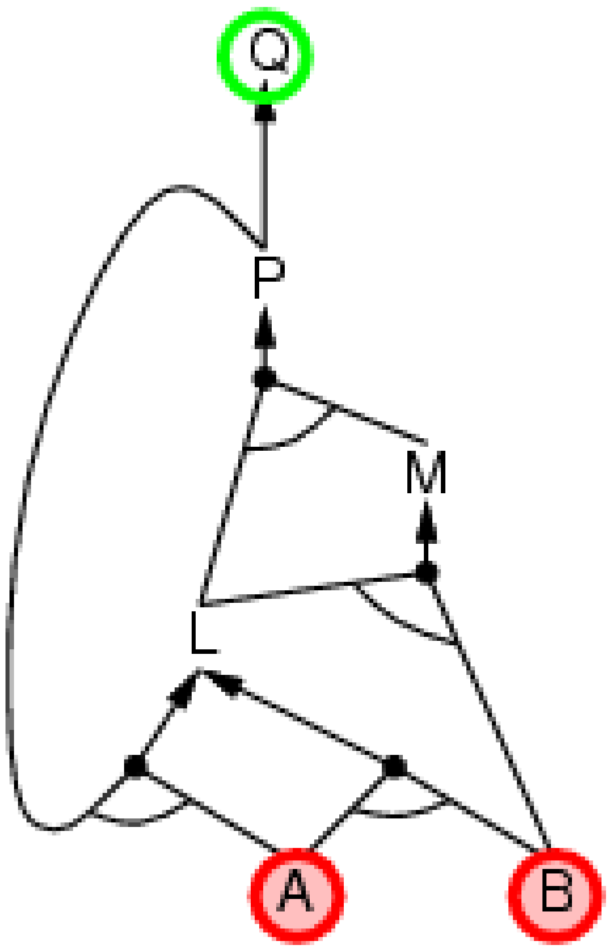
- check if *q* is known already, or
- prove by BC all premises of some rule concluding *q*
- Hence BC maintains a stack of sub-goals that need to be proved to get to q.

Avoid loops: check if new sub-goal is already on the goal stack

Avoid repeated work: check if new sub-goal

1. has already been proved true, or
2. has already failed

Backward chaining example



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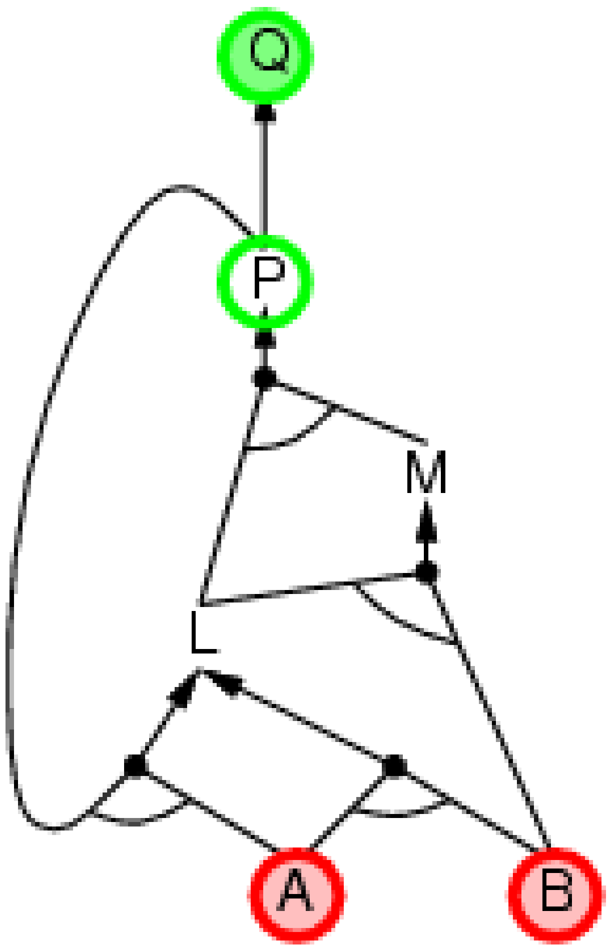

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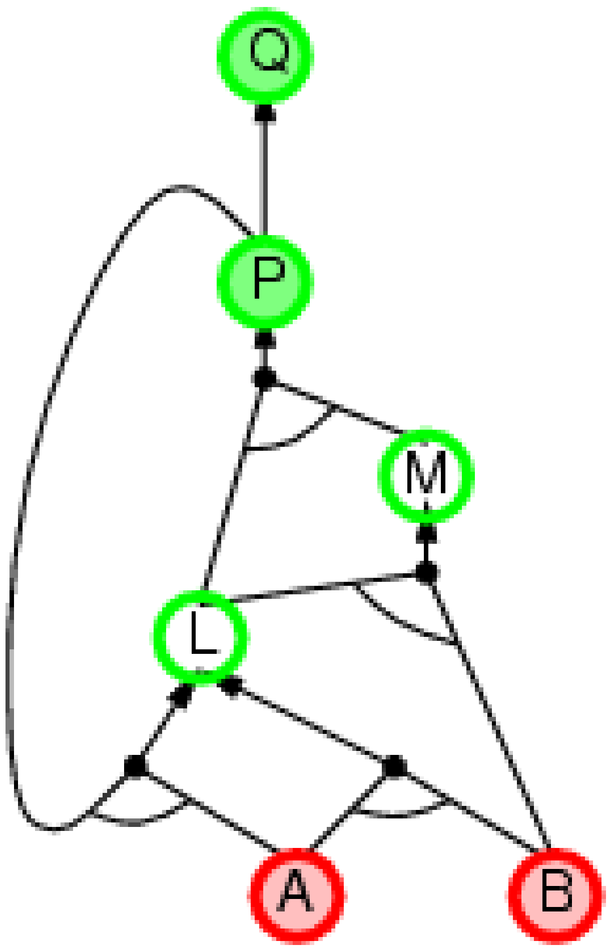
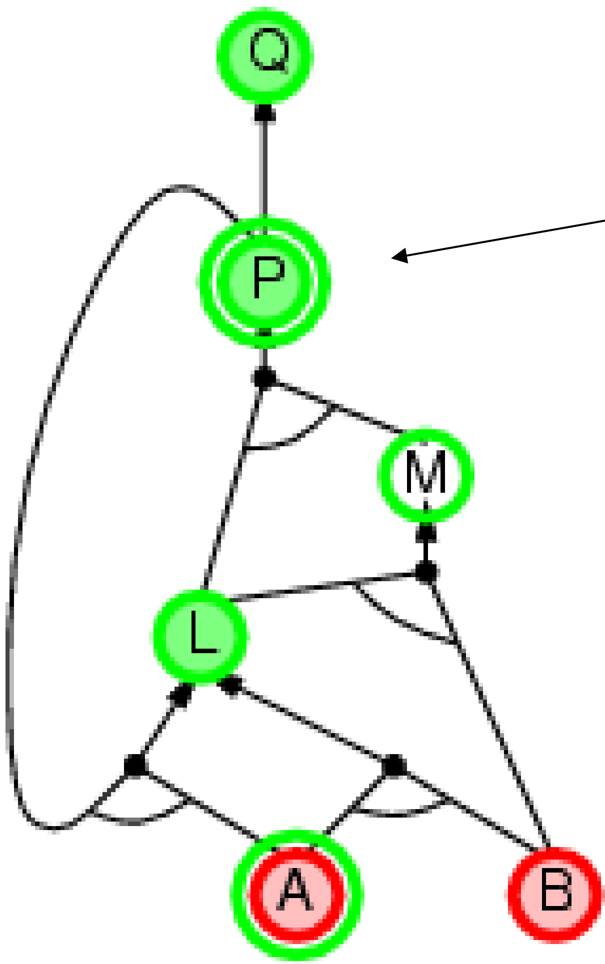
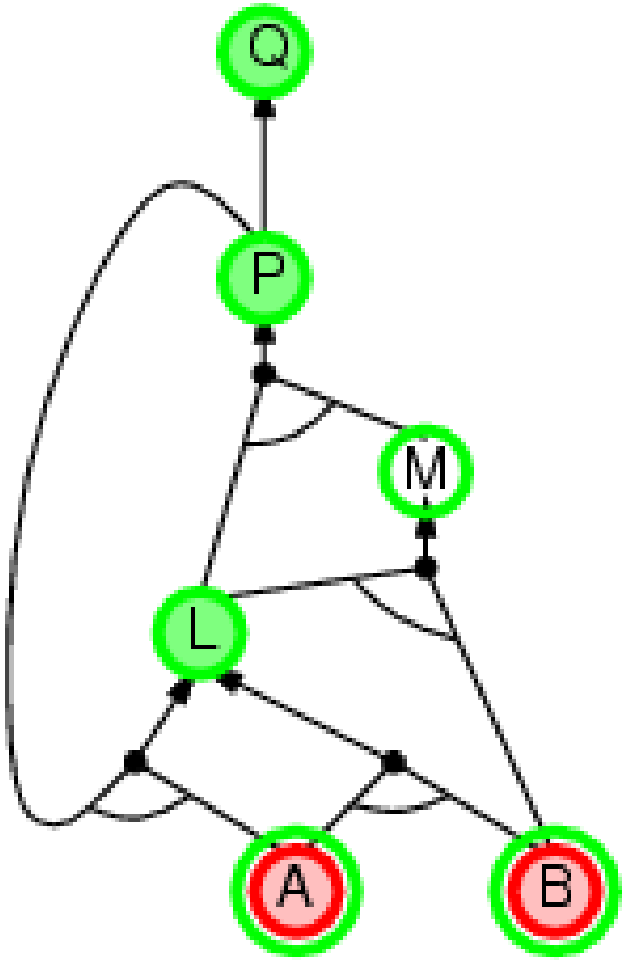


Backward chaining example



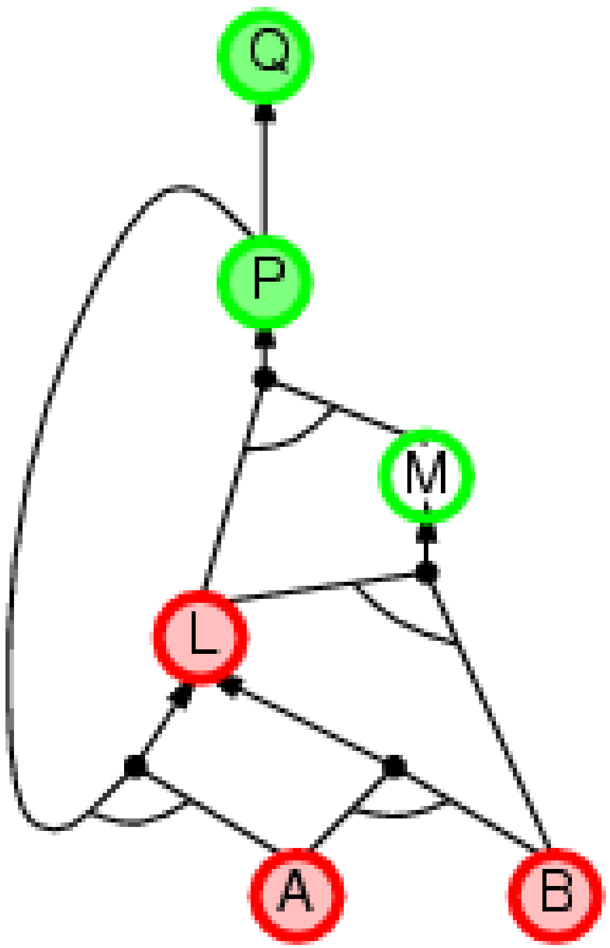
we need P to prove
L and L to prove P.

Backward chaining example

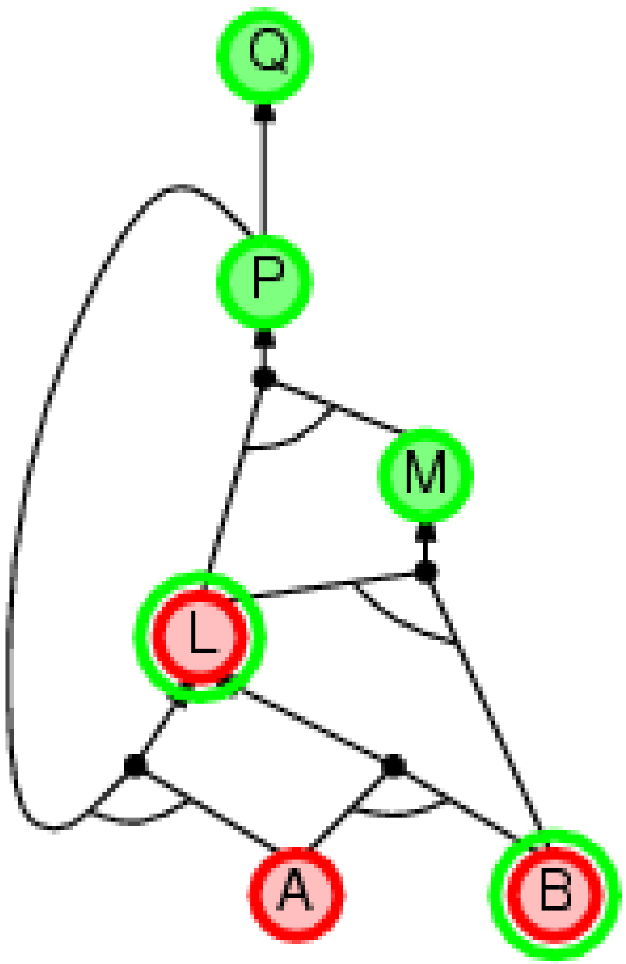


As soon as you can move forward, do so.

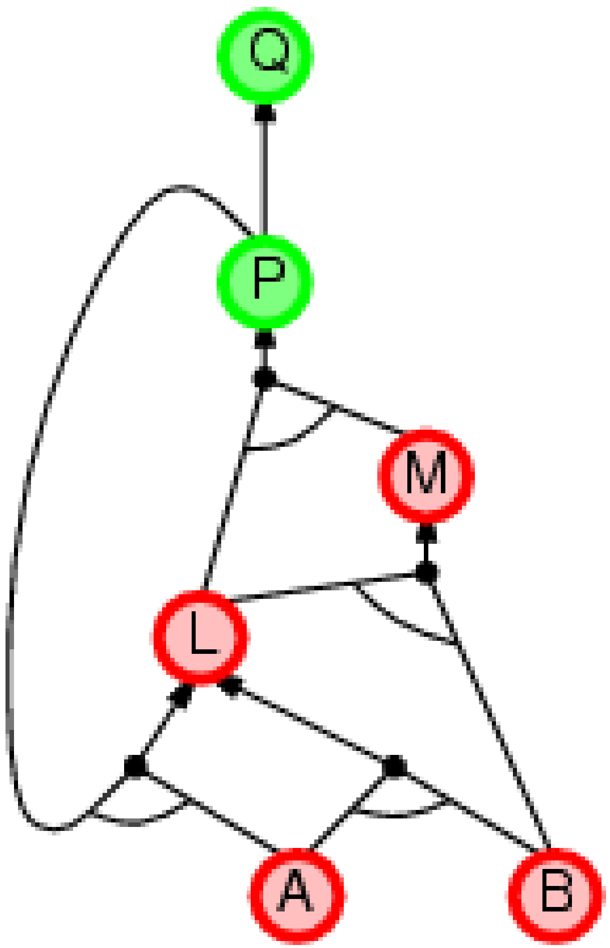
Backward chaining example



Backward chaining example



Backward chaining example



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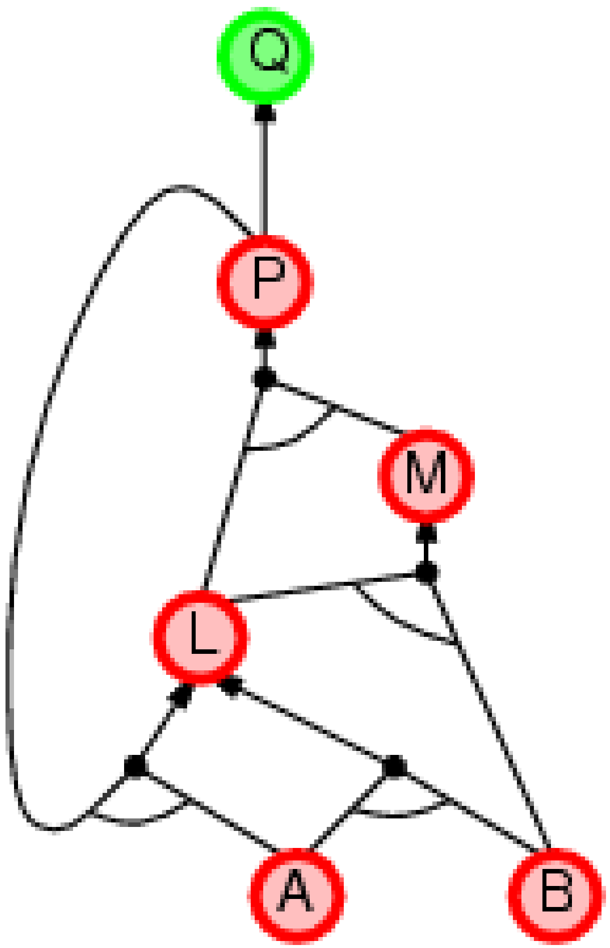

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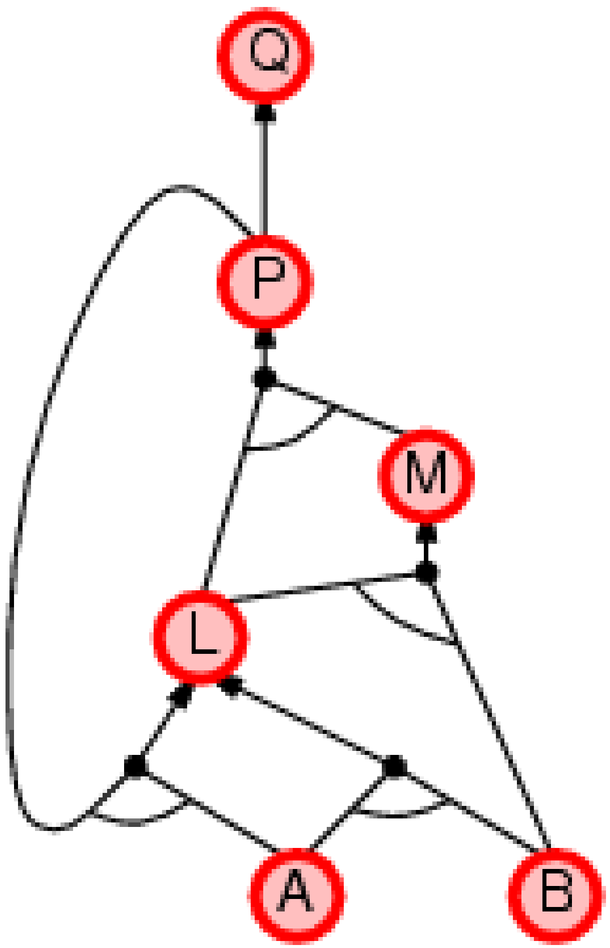


Forward vs. backward chaining

- FC is data-driven, automatic, unconscious processing,
  - e.g., object recognition, routine decisions
- May do lots of work that is irrelevant to the goal
- BC is goal-driven, appropriate for problem-solving,
  - e.g., Where are my keys? How do I get into a PhD program?
- Complexity of BC can be much less than linear in size of KB

Hard satisfiability problems

- Consider *random* 3-CNF sentences. e.g.,

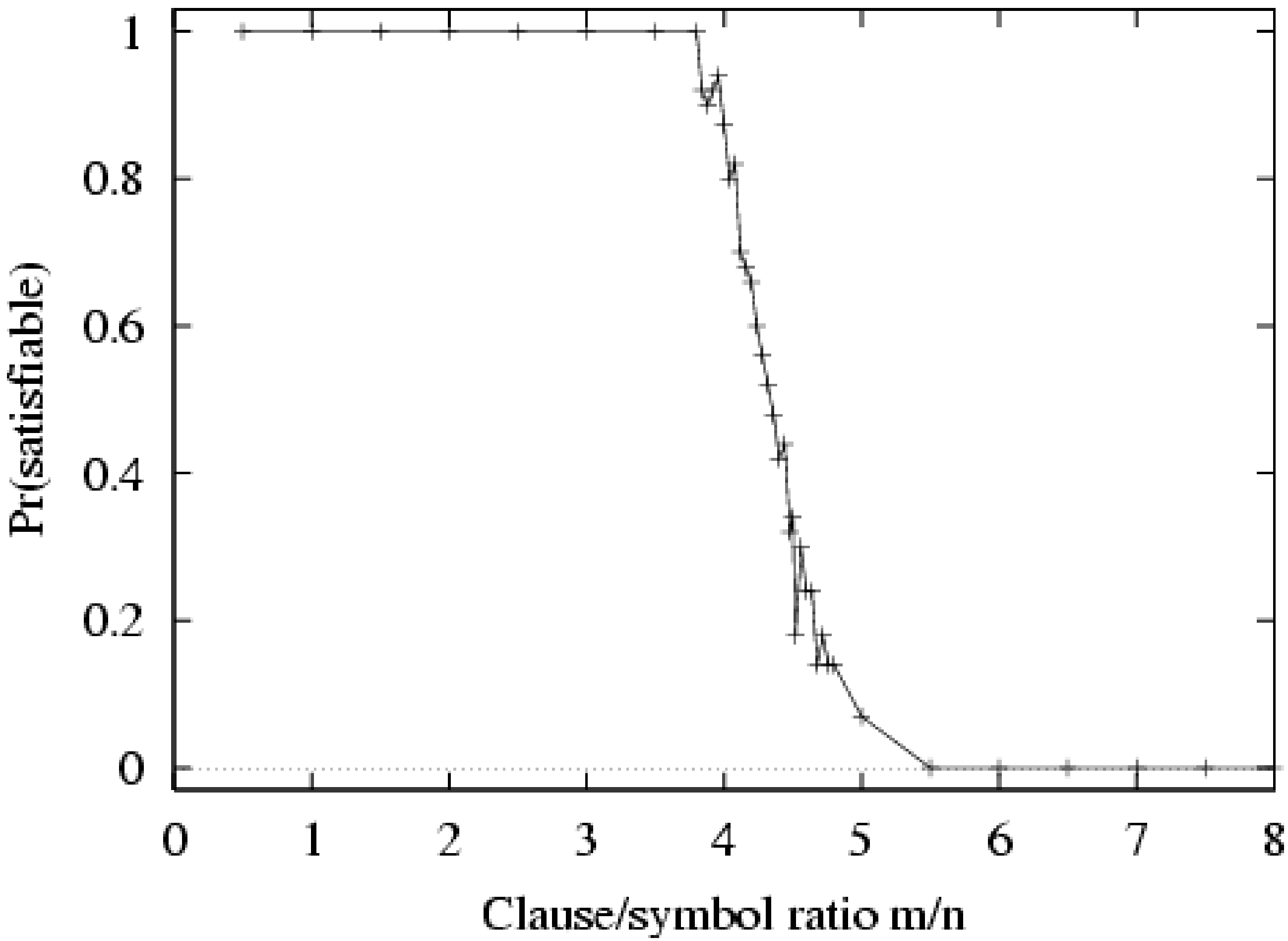
  $(\neg D \lor \neg B \lor C) \land (B \lor \neg A \lor \neg C) \land (\neg C \lor \neg B \lor E) \land (E \lor \neg D \lor B) \land (B \lor E \lor \neg C)$

  m = number of clauses (5)

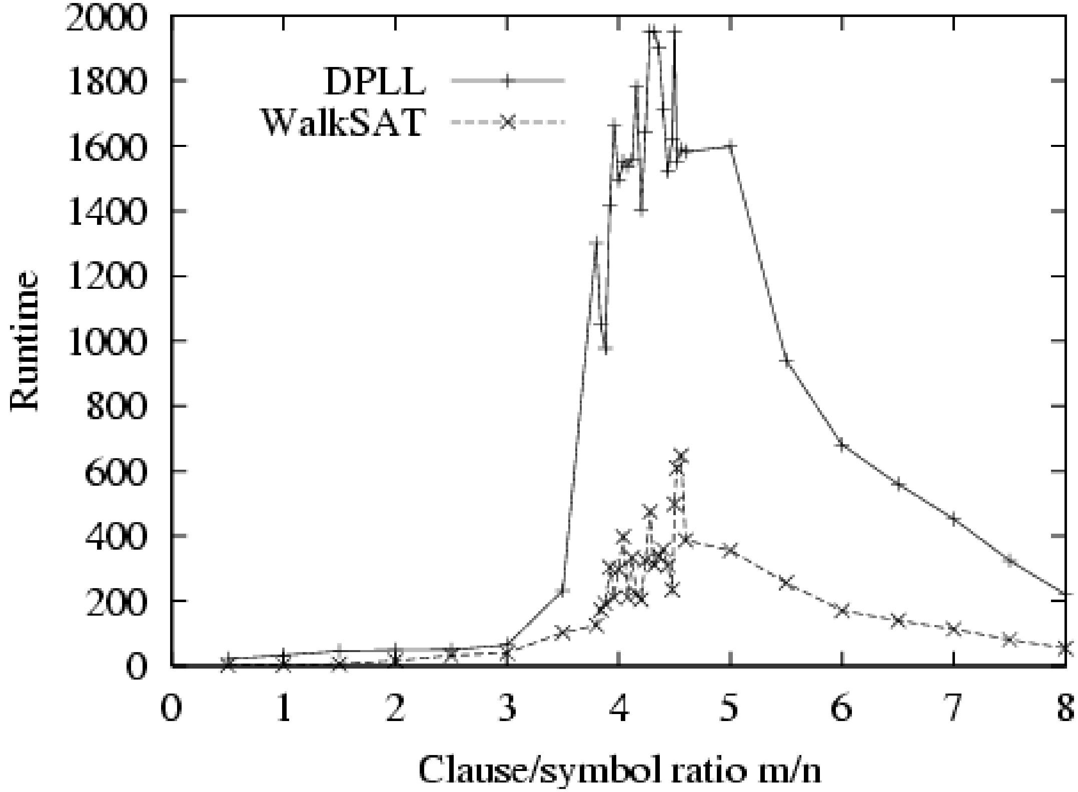
  n = number of symbols (5)

  – Hard problems seem to cluster near $m/n = 4.3$ (critical point)

Hard satisfiability problems



Hard satisfiability problems



- Median runtime for 100 satisfiable random 3-CNF sentences, *n* = 50

Hardness of CSPs

- $x_1 \ldots x_n$ discrete, domain size d: $O(d^n)$ configurations
- "SAT": Boolean satisfiability: d=2
  - The first known NP-complete problem
- "3-SAT"
  - Conjunctive normal form (CNF)
  - At most 3 variables in each clause:

    $$(x_1 \vee \neg x_7 \vee x_{12}) \wedge (\neg x_3 \vee x_2 \vee x_7) \wedge \ldots$$

  - Still NP-complete

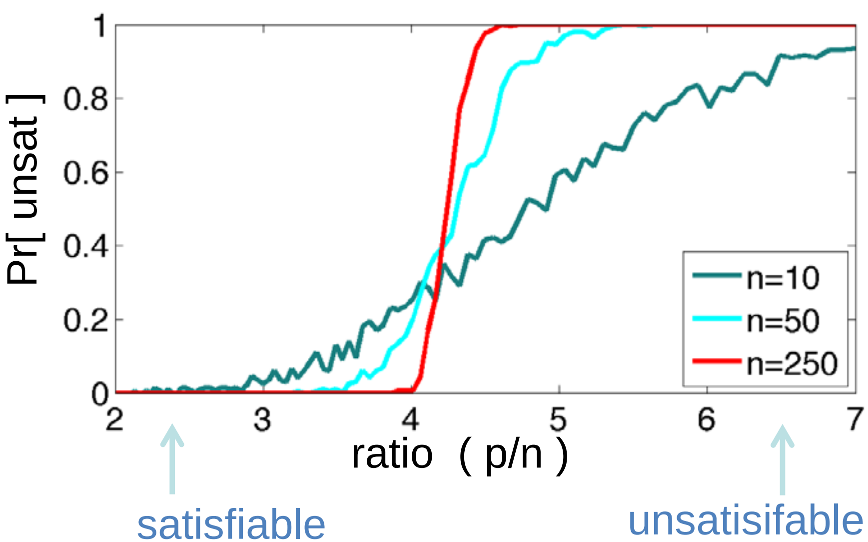
CNF clause: rule out one configuration

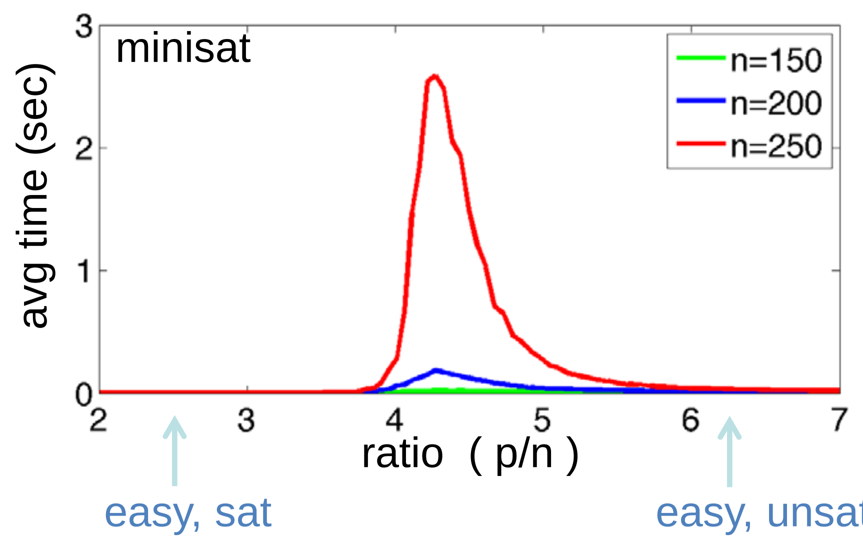
- How hard are "typical" problems?

Hardness of random CSPs

- Random 3-SAT problems:
  - n variables, p clauses in CNF: $(x_1 \vee \neg x_7 \vee x_{12}) \wedge (\neg x_3 \vee x_2 \vee x_7) \wedge \ldots$
  - Choose any 3 variables, signs uniformly at random
  - What's the probability there is **no** solution to the CSP?

  - Phase transition at (p/n) ¼ 4.25
  - "Hard" instances fall in a very narrow regime around this point!





Hardness of random CSPs

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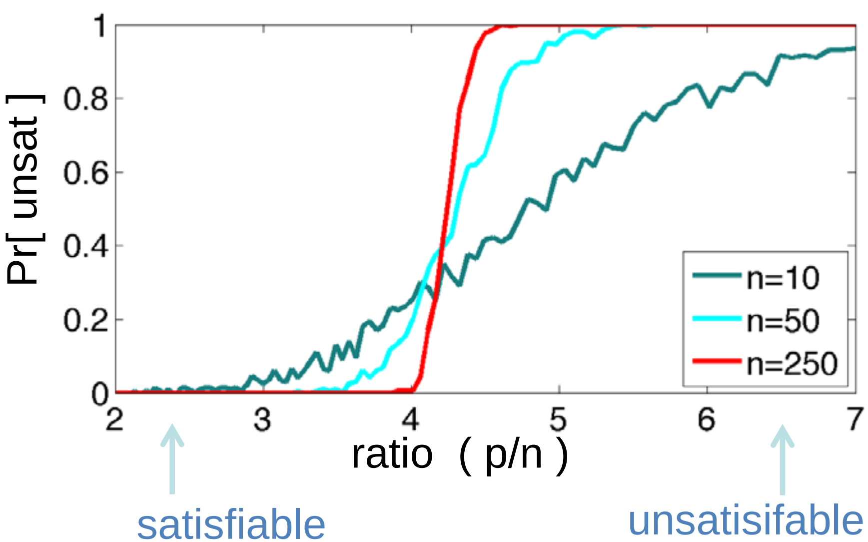

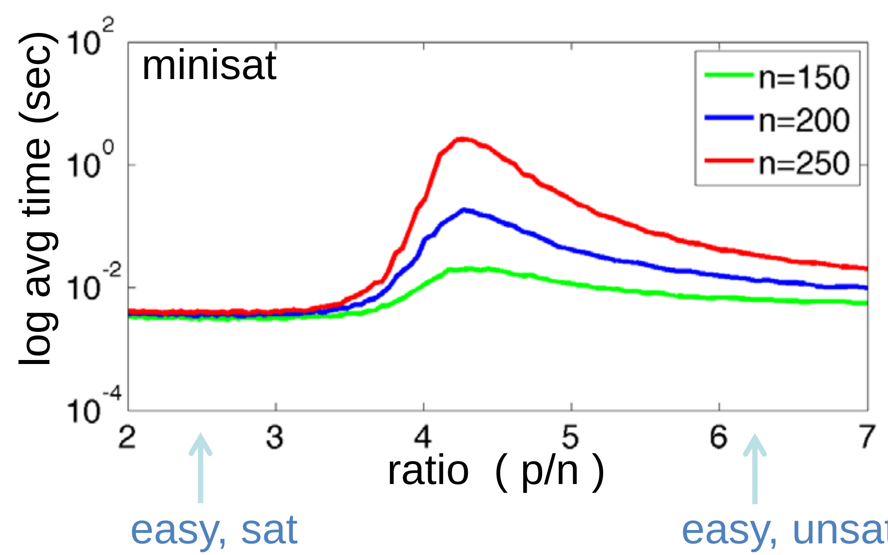

Common Sense Reasoning

Example, adapted from Lenat

You are told: John drove to the grocery store and bought a pound of noodles, a pound of ground beef, and two pounds of tomatoes.

- Is John 3 years old?
- Is John a child?
- What will John do with the purchases?
- Did John have any money?
- Does John have less money after going to the store?
- Did John buy at least two tomatoes?
- Were the tomatoes made in the supermarket?
- Did John buy any meat?
- Is John a vegetarian?
- Will the tomatoes fit in John's car?

- Can Propositional Logic support these inferences?

Summary

- Logical agents apply inference to a knowledge base to derive new information and make decisions
- Basic concepts of logic:
  - syntax: formal structure of sentences
  - semantics: truth of sentences wrt models
  - entailment: necessary truth of one sentence given another
  - inference: deriving sentences from other sentences
  - soundness: derivations produce only entailed sentences
  - completeness: derivations can produce all entailed sentences
- Resolution is complete for propositional logic.
  Forward and backward chaining are linear-time, complete for Horn clauses
- Propositional logic lacks expressive power