Intro Linear Algebra 3A: final

Monday June 12 2017, 10:30–12:30 pm

There are 6 exercises, worth a total of $100 = 12 + 20 + 20 + 12 + 16 + 20$ points.
Non-graphical calculators allowed. No books or notes allowed.
Provide computations and or explanations, unless stated otherwise.

Name:

Student ID:

Exercise 1 (11 pts)
Find the equation $y = \beta_0 + \beta_1 x$ of the least-squares line that best fits the data points $(2, 3)$, $(3, 2)$, $(5, 1)$ and $(6, 0)$.

Solution
$y = 4.3 - 0.7x$.

Exercise 2 ($20 = 6 + 6 + 4 + 4$ pts)
For $x \in \mathbf{R}$ consider the matrix

$$A_x = \begin{bmatrix} 1 & 1 & -1 & x \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & -2 & 2 & 3 \end{bmatrix}.$$

(a) Compute the characteristic polynomial of A_x and show that 1 and 2 are the only eigenvalues.
(b) Compute a basis for each eigenspace of A_x when $x = -1$.
(c) Show that A_x is diagonalizable when $x = -1$, and find an invertible matrix P and a diagonal matrix D such that $A_{-1} = PDP^{-1}$.
(d) For which values of x is A_x diagonalizable?

Solution
(a) $(1 - \lambda)^3(2 - \lambda)$.
(b) E_1 has basis $\{[1, 0, 0, 0]^T, [0, 1, 1, 0]^T, [0, 1, 0, 1]^T\}$. E_2 has basis $\{[-1.0, -1, 2]^T\}$.
(c)

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

(d) Dimension of E_1 is 2 except when $x = -1$. Hence onlly diagonalizable when $x = -1$.

Exercise 3 ($20 = 3 + 1 + 5 + 2 + 4 + 2 + 3$ pts)
Let

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 8 \end{bmatrix}.$$

Consider the subspace $H = \mathrm{Span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} \subseteq \mathbf{R}^5$. Let $\mathbf{v} = [1, 3, 2, 2, 2]^T$.
(a) Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a basis of H.
(b) What is the dimension of H?
(c) Compute an orthogonal basis for H.
(d) Compute an orthonormal basis for H.
(e) Compute the orthogonal projection of $\mathbf{v}$ onto H.
(f) Find the distance between $\mathbf{v}$ and H.
(g) Find a basis of $H^{\perp}$.

Solution
(a) Easy check (folows also from c)
(b) 3
(c) $\{[1, 1, 1, 1, 1]^T, [0, 0, 1, -1, 0]^T, [-2, -2, -1, -1, 6]^T\}$.
(d) $\{1/\sqrt{5}[1, 1, 1, 1, 1]^T, 1/\sqrt{2}[0, 0, 1, -1, 0]^T, 1/\sqrt{46}[-2, -2, -1, -1, 6]^T\}$.
(e) $2[1, 1, 1, 1, 1]^T$

(f) the distance vector is $[-1,1,0,0,0]^T$, and the distance is $\sqrt{2}$.
(g) $\{[-1,1,0,0,0]^T, [7,0,-4,-4,1]^T\}$.

Exercise 4 ($12 = 3 + 3 + 1 + 3 + 2$ pts)
For $x \in \mathbf{R}$ consider the matrix

$$A_x = \begin{bmatrix} 1 & 2 & 0 & 0 & -3 & 1 \\ 0 & 0 & 1 & 0 & x & 1 \\ 0 & 0 & 0 & x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Compute a basis of the null space of A_x when $x = 2$.
(b) For each x, compute the rank of A_x.
(c) For which x is A_x invertible?
(d) Is $\{[1,0,0,0]^T, [1,1,0,0]^T, [1,1,1,0]^T\}$ a basis for the column space of A_x when $x = 3$?
(e) Find a nonzero square matrix B with $B^2 = 0$.

Solution
(a) $\{[-2,1,0,0,0,0]^T, [3,0,-2,0,1,0]^T, [-1,0,-1,0,0,1]^T\}$
(b) $x \neq 0$, dimension is 3, for $x = 0$, the dimension is 2.
(c) Not square, never invertible.
(d) Yes, they column space is equal to $\{[x,y,z,0]^T : x,y,z \in \mathbf{R}\}$, and so is the span of the vectors.
(e)

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Exercise 5 ($16 = 6 + 6 + 4$ pts)
(a) Use row operations to compute the inverse of

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}.$$

(b) Use Cramer's rule (determinants) to compute the inverse of

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}.$$

(c) Let A be an $n \times n$ matrix. Assume that 1 is an eigenvalue of $A^T + I_n$. Show that A is not invertible.

Solution:
(a)

$$\begin{bmatrix} -\frac{1}{5} & -\frac{8}{5} & \frac{3}{5} \\ \frac{2}{5} & \frac{1}{5} & -\frac{1}{5} \\ 0 & 1 & 0 \end{bmatrix}.$$

(b)

$$\begin{bmatrix} -\frac{1}{3} & -\frac{5}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \\ 0 & 1 & 0 \end{bmatrix}.$$

(c) Let $\mathbf{x}$ nonzero with $(A^T + I)\mathbf{x} = 1\mathbf{x}$. Then $A^T\mathbf{x} = 0$. So A^T is not invertible. So A is not invertible.

Exercise 6 (20 pts)
True of false? No explanation required. Each question is worth 2 points.

(1) Let A be an $m \times n$ matrix and let $\mathbf{b} \in \mathbf{R}^m$. Then the equation $A^T A\mathbf{x} = A^T\mathbf{b}$ is always consistent.
(2) Let $H \subseteq \mathbf{R}^n$ be a subspace. Then the orthogonal projection onto H is a linear map from $\mathbf{R}^n$ to $\mathbf{R}^n$.
(3) Let $\mathbf{x} \in \mathbf{R}^n$ nonzero. Then there is an $n \times n$ matrix A with $\mathbf{x}$ as eigenvector.
(4) Let A be an $n \times n$ matrix which is not diagonalizable. Let P be an invertible matrix. Then PAP^{-1} is not diagonalizable.
(5) Let A be an $n \times n$ matrix with orthogonal columns. Then $A^T A = I_n$.
(6) The parallelogram determined by the points $(-1, 1), (0, 1), (2, 3), (3, 3)$ has area 3.
(7) The set $\{(x, y) \in \mathbf{R}^2 : xy = 0\}$ is a subspace of $\mathbf{R}^2$.
(8) Let $H \subseteq \mathbf{R}^7$ be a subspace. Let $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_7\}$ with $\mathrm{Span}(S) = H$. Then S is a basis of H.
(9) Let A, B be $n \times n$ matrices. Assume that A is invertible. Then $(A^2)^T(A+B)B^2$ is invertible.
(10) Let A, B be $n \times n$ matrices with the same reduced row echelon form. Then A and B have the same characteristic polynomial.

Solution:
(1) True.
(2) True.
(3) True.
(4) True
(5) False.
(6) False.
(7) False.
(8) False.
(9) False.
(10) False.