Intro Linear Algebra 3A: midterm 2

Monday February 27 2017, 00:00 – 00.50 pm

There are 4 exercises, worth a total of $100 = 35 + 16 + 21 + 28$ points.
Non-graphical calculators allowed. No books or notes allowed.
Provide computations and or explanations, unless stated otherwise.

Name:

Student ID:

Exercise 1 (35 = 5 + 5 + 10 + 10 + 5 pts) For $s \in \mathbf{R}$ consider the matrix

$$A_s = \begin{bmatrix} 2 & 1 & 2 \\ 2 & s & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

(a) Compute A_s^2 when $s = 0$.
(b) For which s is A_s invertible?
(c) Compute the inverse of A_s by using elementary row operations when $s = 1$.
(d) Compute the inverse of A_s by using Cramer's rule (determinants) when $s = 2$.
(e) (Hard) For which s is the upper left entry of A_s^{-1} equal to 5?

Solution:
(a)

$$A_0^2 = \begin{bmatrix} 6 & 4 & 7 \\ 4 & 3 & 5 \\ 2 & 1 & 2 \end{bmatrix}.$$

(b) $\det(A_s) = 2s$, so invertible when $s \neq 0$.

(c) $\begin{bmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ -1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$.

(d) $\begin{bmatrix} \frac{1}{4} & \frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

(e) The first entry of the inverse, by Cramer's rule, is equal to $(s-1)/2s$. So $s = -1/9$ is the only solution.

Exercise 2 (16 = 7 + 7 + 2 pts)
Consider the matrix

$$M = \begin{bmatrix} 1 & 2 & -1 & 1 & -1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 3 & -2 & -1 & 1 \\ 2 & 3 & -1 & 2 & 0 \end{bmatrix}$$

with reduced row echelon form

$$E = \begin{bmatrix} 1 & 0 & 1 & 0 & 5 \\ 0 & 1 & -1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Compute a basis of the null space of M.
(b) Compute a basis of the column space of M.
(c) What is the rank of M?

Solution:
(a) $\{[-1, 1, 1, 0, 0]^T, [-5, 2, 0, 2, 1]^T\}$.
(b) Take first, second and fourth column of M.
(c) 3.

Exercise 3 (21 = 7 + 14 pts)
Consider the matrix

$$A = \begin{bmatrix} 3 & -1 & 1 \\ 2 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) Compute the characteristic polynomial of A and show that $1, 1, 2$ are the eigenvalues of A (counted with multiplicities).
(b) For each eigenvalue of A, compute a basis for its corresponding eigenspace.

Solution:
(a) $-(\lambda-1)^2(\lambda-2)$ is the characteristic polynomial. Eigenvalues follow easily.
(b) $\lambda = 1$: $\{[-1,0,2]^T, [1,2,0]^T\}$. For $\lambda = 2$, one gets $\{[1,1,0]^T\}$.

Exercise 4 (28 pts)
True or false? **No** explanation required. Points is $-12 + 4 \cdot \#$correct.

(1) Let A, B be matrices of the same size. Then one has $(A+B)^T = B^T + A^T$.
(2) Let A be an $n \times m$ matrix with column space $\mathbf{R}^n$. Then A is invertible.
(3) Any plane in $\mathbf{R}^3$ is a subspace of $\mathbf{R}^3$.
(4) Any subspace of $\mathbf{R}^n$ has a unique basis.
(5) Let A be a square matrix which is not invertible and let $c \in \mathbf{R}$. Then one has $\det(cA) = c\det(A)$.
(6) Let A, B be two square matrices of the same size. One has $\det(A+B) = \det(A) + \det(B)$.
(7) Let A be a 3×3 matrix with determinant -1. Then the linear transformation defined by A preserves volumes.
(8) Let A be a square matrix with eigenvalue 3. Then A^5 is a matrix with eigenvalue 3^5.
(9) If A, B be $n \times n$ matrices. If $\mathbf{x}$ is an eigenvector of both A and B, then $\mathbf{x}$ is an eigenvector of AB.
(10) Let A, B be $n \times n$ matrices. Then one has $(A+B)^2 = A^2 + 2AB + B^2$.

Solution:
(1) True.
(2) False, not square.
(3) False, does not contain 0 in general.
(4) False, many bases exist.
(5) True, both are 0.
(6) False.
(7) True.
(8) True.
(9) True.
(10) False.