Intro Linear Algebra 3A: final

Monday March 20 2017, 1:30 – 3.30 pm

There are 5 exercises, worth a total of $100 = 20 + 25 + 20 + 15 + 20$ points.
Non-graphical calculators allowed. No books or notes allowed.
Provide computations and or explanations, unless stated otherwise.

Name:

Student ID:

Exercise 1 (20 = 5 + 5 + 5 + 5 pts)
For $x \in \mathbf{R}$ consider the matrix

$$A_x = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 4 & 1 \\ 1 & x & 4 \end{bmatrix}.$$

Set $B = A_x$ where $x = 1$.
(a) Compute the characteristic polynomial of B and show that 3 and 5 are the eigenvalues of B.
(b) For each eigenvalue of B, compute a basis of the corresponding eigenspace of B.
(c) Is B diagonalizable? If so, find an invertible matrix P and a diagonal matrix D with $B = PDP^{-1}$.
(d) (hard) For which x is A_x diagonalizable over $\mathbf{R}$?

Solution:
(a) $-(\lambda-3)^2(\lambda-5)$.
(b) E_3 has basis $\{[-1,0,1]^T, [-1,1,0]^T\}$. E_5 has basis $\{[0,1,1]^T\}$.
(c) Yes, $D = \mathrm{diag}(3,3,5)$ and

$$P = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

(d) The characteristic polynomial if $(3-\lambda)\cdot\big((4-\lambda)^2 - x\big)$. Over $\mathbf{R}$: if $x < 0$, then we don't see all eigenvalues, not diagonalizable. If $x > 0$ then it has distinct eigenvalues unless $x = 1$, but in the latter case we have shown it is diagonalizable. We only have to check $x = 0$. In that case the matrix turns out not to be diagonalizable. Hence the matrix is diagonalizable for $x > 0$.

Exercise 2 (25 = 3 + 4 + 4 + 4 + 5 + 5 pts)
For $x \in \mathbf{R}$ consider the matrix

$$A_x = \begin{bmatrix} 1 & 0 & x \\ 1 & x & x \\ 1 & 3 & 1. \end{bmatrix}.$$

(a) Compute A_x^2 when $x = 1$.
(b) Show that A_x is invertible for $x \neq 0, 1$.
(c) For $x = 0$ compute a basis for the null space and column space of A_x.
(d) For $x = 1$, compute the rank and the dimension of the null space of A_x .
(e) Compute A_x^{-1} when $x = 2$.
(f) (subtle) Set $\mathbf{v} = [0,1,3]^T$. Consider the equation $A_x[x_1,x_2,x_3]^T = \mathbf{v}$. For which x is there a solution with $x_3 = 1$?

Solution:
(a)

$$\begin{bmatrix} 2 & 3 & 2 \\ 3 & 4 & 3 \\ 5 & 6 & 5 \end{bmatrix}.$$

(b) $\det(A_x) = x - x^2 = x(1-x)$. So invertible when $x \neq 0, 1$.
(c) Column space: $\{[1,1,1]^T, [0,0,3]^T\}$. Null space $\{[0,1,-3]^T\}$.
(d) Rank 2, dimension null space 1.

(e)

$$\begin{bmatrix} 2 & -3 & 2 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{3}{2} & -1 \end{bmatrix}.$$

(f) If $x \neq 0, 1$ we can use Cramer's rule $1 = (3x-3)/(x-x^2)$, and this has solution $x = -3, 1$. The valid solution is $x = -3$. We need to check the non invertible cases separately. It turns out that $x = -3, 1$ are the only cases in the end.

Exercise 3 (20 = 8 + 4 + 3 + 5 pts)
Let

$$H = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \\ 1 \end{bmatrix} \right\} \subset \mathbf{R}^4.$$

(a) Find an orthonormal basis of H.
(b) Compute the orthogonal projection of $[0, 2, 2, 4]^T$ on H.
(c) Compute the distance from $[0, 2, 2, 4]^T$ to H.
(d) Find a basis of $H^\perp$.

Solution:
(a) Gram-Schmidt: $\{1/\sqrt{2}[1, 0, 1, 0]^T, 1/2[1, 1, -1, 1]^T, 1/\sqrt{6}[-1, 2, 1, 0]^T\}$.
(b) $[1, 3, 1, 1]^T$.
(c) $\sqrt{12}$.
(d) $\{[-1, -1, 1, 3]^T\}$.

Exercise 4 (15 = 3 + 3 + 6 + 3)
Consider the linear map $T : \mathbf{R}^3 \to \mathbf{R}^3$ given by

$$T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x + y + z \\ -x - z \\ -x - y \end{bmatrix}.$$

(a) Find the standard matrix M of T.
(b) Show that $M^2 = M$.
(c) Show that M is diagonalizable.
(d) Is there a subspace $W \subseteq \mathbf{R}^3$ such that $T = \text{Proj}_W$?

Solution:
(a)

$$\begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}.$$

(b) Easy computation.
(c) (Any projection is diagonalizable) Characteristic polynomial is $-(\lambda - 1)^2\lambda$. The matrix is diagonalizable since E_1 has dimension 2.
(d) Null space is span of $[-1, 1, 1]^T$. This vector is not orthogonal to $[2, -1, -1]$. So, the answer is no.

Exercise 5 (20 pts)
True of false? No explanation required. Points = $3 \times$ #correct $- 10$.

(1) Let A be an 6×5 matrix. Assume that the rank of A is 3. Then the dimension of the null space of A is 3.
(2) Let A, B be two diagonalizable matrices of the same size. Then AB is diagonalizable.
(3) Let H be a subspace of $\mathbf{R}^n$. Then there is an $n \times n$ matrix A such that the column space of A is equal to H.
(4) Let A be an $n \times n$ matrix and assume that $\mathcal{B}$ is a basis of $\mathbf{R}^n$ of eigenvectors of A. Then $[A]_{\mathcal{B}}$ is a diagonal matrix.
(5) If two matrices have the same characteristic polynomial, then they are similar.
(6) Let H_1 and H_2 be subspaces of $\mathbf{R}^n$. Then the intersection $H_1 \cap H_2 = \{x \in \mathbf{R}^n : x \in H_1,\ x \in H_2\}$ is a subspace of $\mathbf{R}^n$.
(7) Let A, B be 3×3 matrices with $\det(A) = -1$, $\det(B) = 2$. Then $\det(A(-B)A^2) = 2$.
(8) Let A be an $n \times n$ matrix. Then there are only finitely many $a \in \mathbf{R}$ such that $A - aI$ is not invertible.
(9) Every square matrix is diagonalizable over the complex numbers.
(10) Let A be an invertible $n \times n$ matrix. Then there are no $\mathbf{x} \in \mathbf{R^n}$ with $A\mathbf{x} = 0$.

Solution:
(1) False, it is 2.
(2) False,

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

(3) True, put a basis in the columns of the matrix and add extra 0 columns.
(4) True.
(5) False.
(6) True.
(7) True.
(8) True, characteristic polynomial has finite degree.
(9) False.
(10) False, always 0 vector.