Systems, System Variables and Models

The definition of a system is quite general.

There are many systems one can think of:

- solar system
- ecosystem
- economic system
- nervous system
- cardiovascular system
- educational system
- operating system (computers)
- etc.

Any collection of elements that are functionally integrated with a common goal is considered a system.

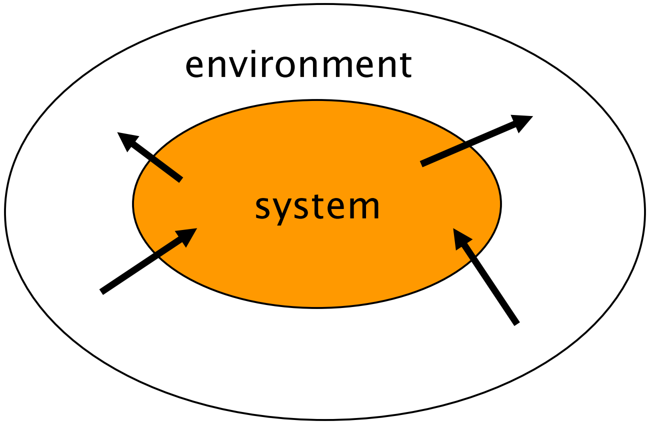
The elements may be physical or abstract.

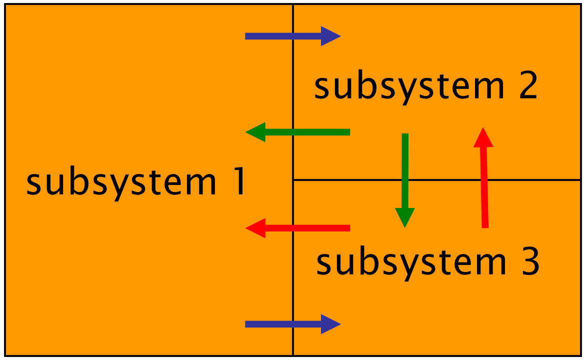
Definition 1 *System is defined as a collection of units (elements, parts, devices, organs), functionally organized to accomplish certain goals by consuming, transforming and exchanging energy, matter and/or information.*

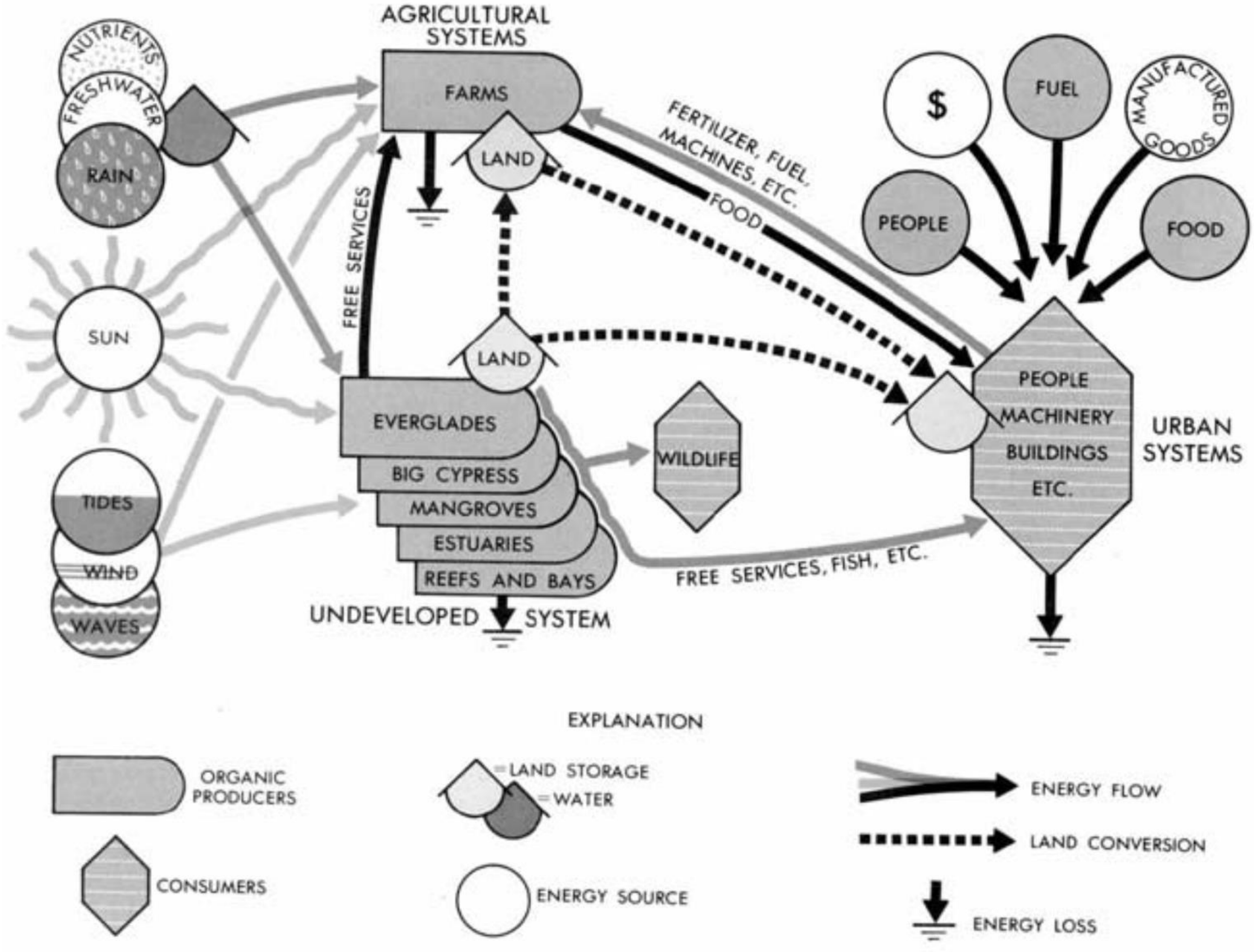
In engineering, systems are typically associated with objects (e.g. machines, electric circuits, tissues, ...) and/or processes (chemical reaction, blood flow, gas exchange, …). It is important to realize that:

1) No system is isolated (a system always interacts with the environment and other systems)

2) A system often comprises other simpler interacting systems called subsystems







AGRICULTURAL SYSTEMS
FARMS
LAND
NUTRIENTS
FRESHWATER
RAIN
SUN
TIDES
WIND
WAVES
FREE SERVICES
FERTILIZER, FUEL, MACHINES, ETC.
FOOD
$
FUEL
MANUFACTURED GOODS
PEOPLE
FOOD
LAND
EVERGLADES
BIG CYPRESS
MANGROVES
ESTUARIES
REEFS AND BAYS
UNDEVELOPED SYSTEM
WILDLIFE
PEOPLE MACHINERY BUILDINGS ETC.
URBAN SYSTEMS
FREE SERVICES, FISH, ETC.
EXPLANATION
ORGANIC PRODUCERS
LAND STORAGE
WATER
ENERGY FLOW
LAND CONVERSION
CONSUMERS
ENERGY SOURCE
ENERGY LOSS

Ecosystem

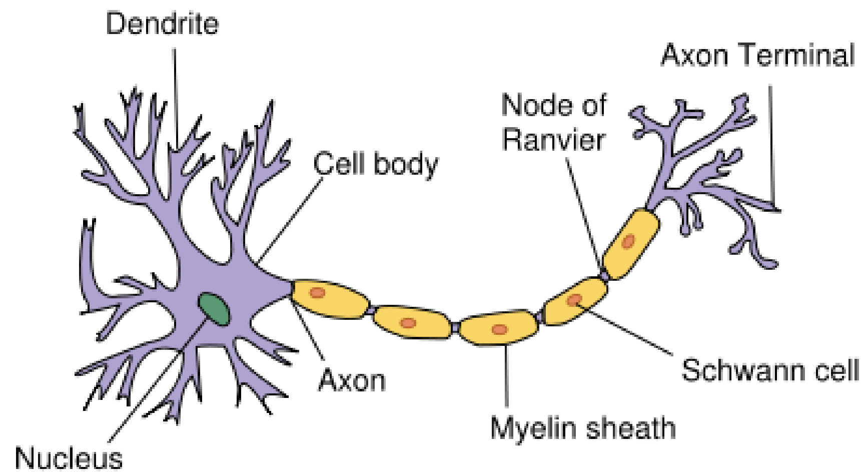
Definition 2 *System variables are physical or abstract quantities that influence or characterize the behavior of the system.*

Examples:
- position
- velocity
- temperature
- inflation rate
- stock market value
- blood pressure
- hormone concentration
- etc.

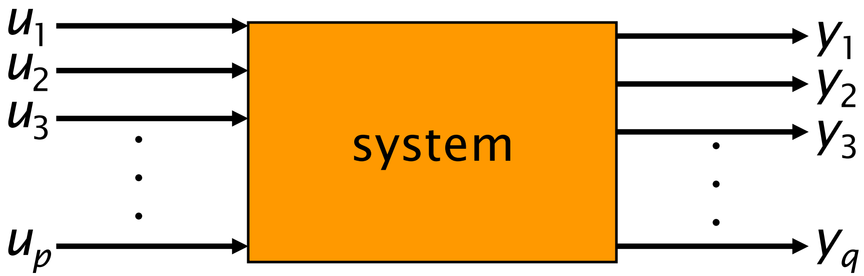
Example 1

System: neuron

Variables: - ion concentration
- membrane current
- membrane potential
- temperature
- etc.



System is typically represented by a schematic called system diagram, and variables are represented by arrows.



External variables that critically affect the function of a system are called <u>input variables</u> or short inputs (cause).

Inputs are denoted by arrows entering the system.

Internal variables that depend on system's behavior are called <u>output variables</u> or just outputs (effect).

Outputs are denoted by arrows leaving the system.

Definition 3 *A system that has a single input and a single output is called **SISO** system* $(p = q = 1)$*. A system that has multiple inputs and/or multiple outputs is called **MIMO** system.*

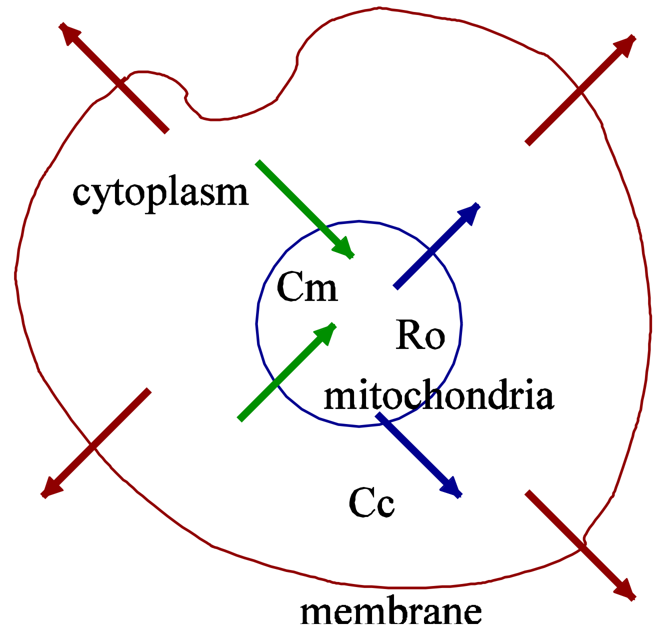
Internal variables that characterize the behavior of a system are called <u>state variables</u> or short states.

Variables that do not change appreciably in time and/or space are called <u>parameter variables</u> or just parameters.

Example 2 Cellular dynamics

System: cell
Input: the rate of synthesis R_0
Output: the concentration C_c
State: the concentration C_m
Parameters: the rate constants K_{12} and K_2

(demo: `cellular_dynamics.m`)



$$V_m\dot{C}_m(t) = R_0(t) - K_{12}[C_m(t) - C_c(t)]$$
$$V_c\dot{C}_c(t) = K_{12}[C_m(t) - C_c(t)] - K_2C_c(t)$$

This nomenclature is somewhat subjective. For whatever reason, I might declare C_m as an output variable, and C_c as a state variable. Here are some general guidelines:

1) Outputs – variables of interest that depend on system's behavior.

2) States – variables that depend on system's behavior, but we are not directly interested in them.

Note: A system variable may play the role of both output and state simultaneously.

Note: Not all system's variables are measurable. For example, state variables are often not measurable.

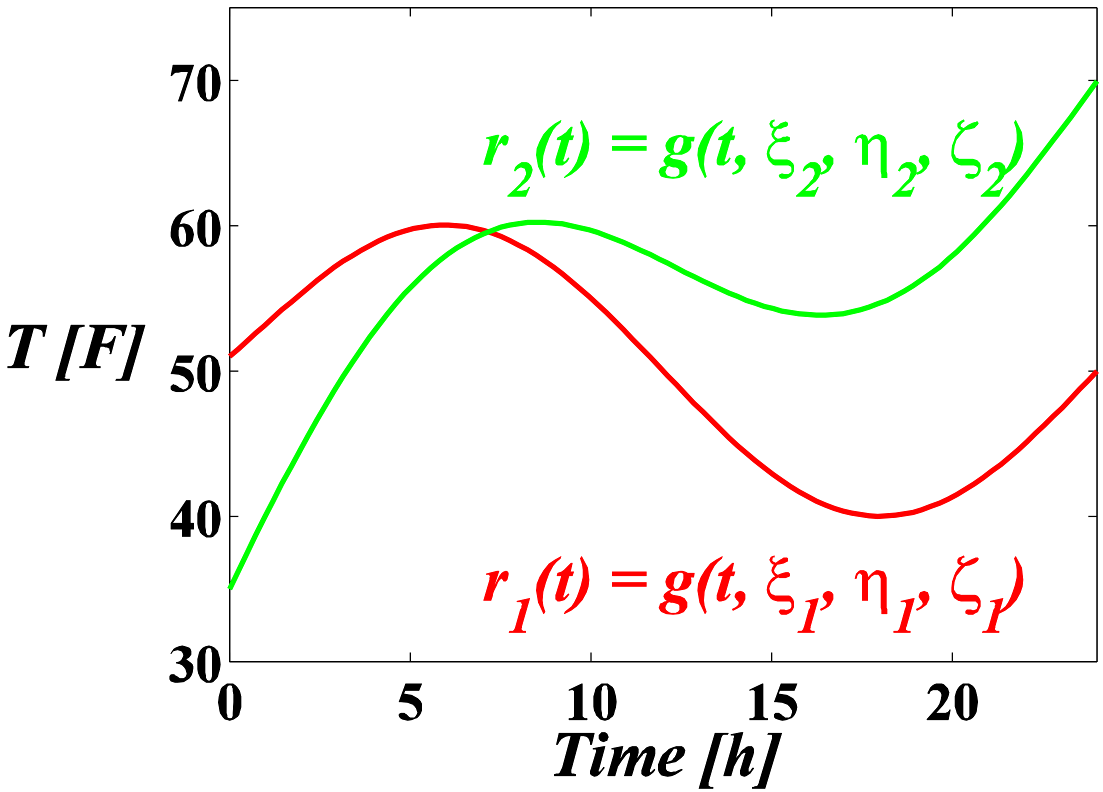
Parameters are rarely measurable. When possible, they are estimated from other (observable) variables. The same stands for unmeasurable (unobservable) variables. These problems are the subject of estimation theory.

In the context of systems theory, estimation is often referred to as system identification.

Generally, system variables are changing in time and space, e.g. $C_c = f(t, \xi, \eta, \zeta)$. These variables are called spatio-temporal variables, and they obey partial differential equations (PDEs).

Example 3

Temperature--two identical thermometers in two different rooms might show different temperature readings over time:



Similarly, variables that only depend on time are called temporal variables. We will see later that temporal variables obey ordinary differential equations (ODEs).

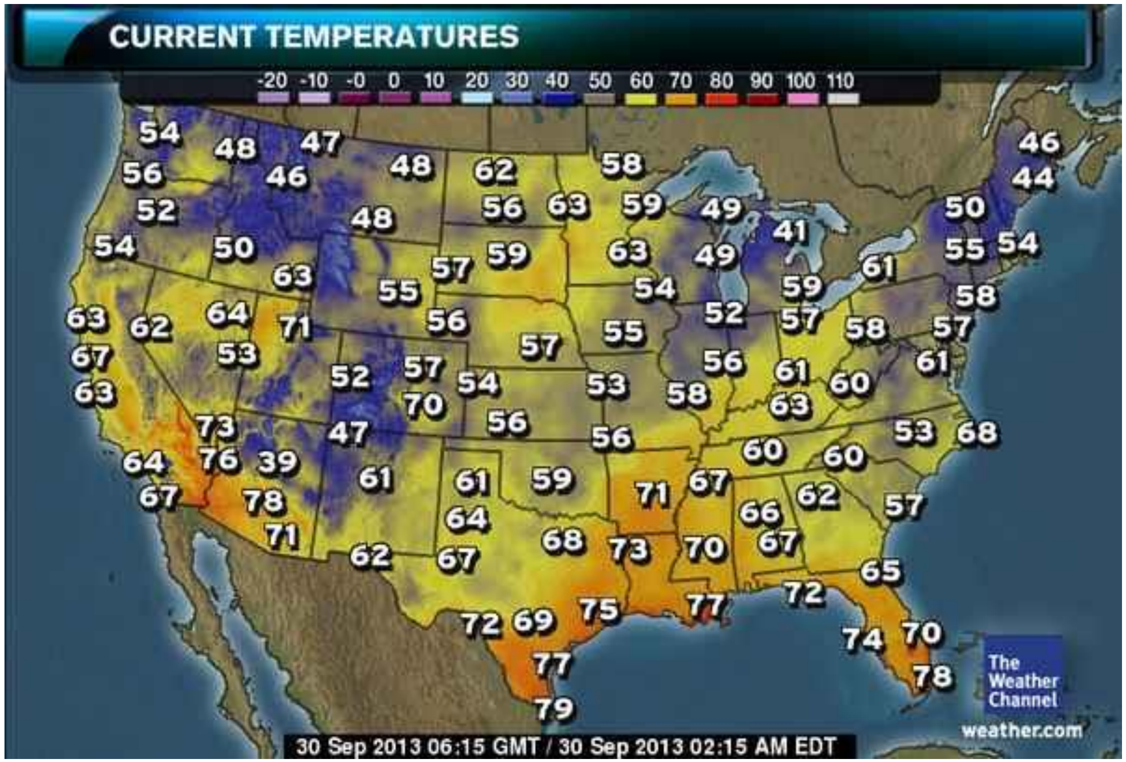
Temporal variables typically arise from spatio-temporal variables when spatial coordinates are fixed.

Example 4 Room temperature T_i at a point (ξ_0, η_0, ζ_0) is an example of a temporal variable, i.e. $T_i = f(t, \xi_0, \eta_0, \zeta_0) = f(t)$.

Proceeding in this fashion, we can fix time t in a spatio-temporal variable to obtain a spatial variable.

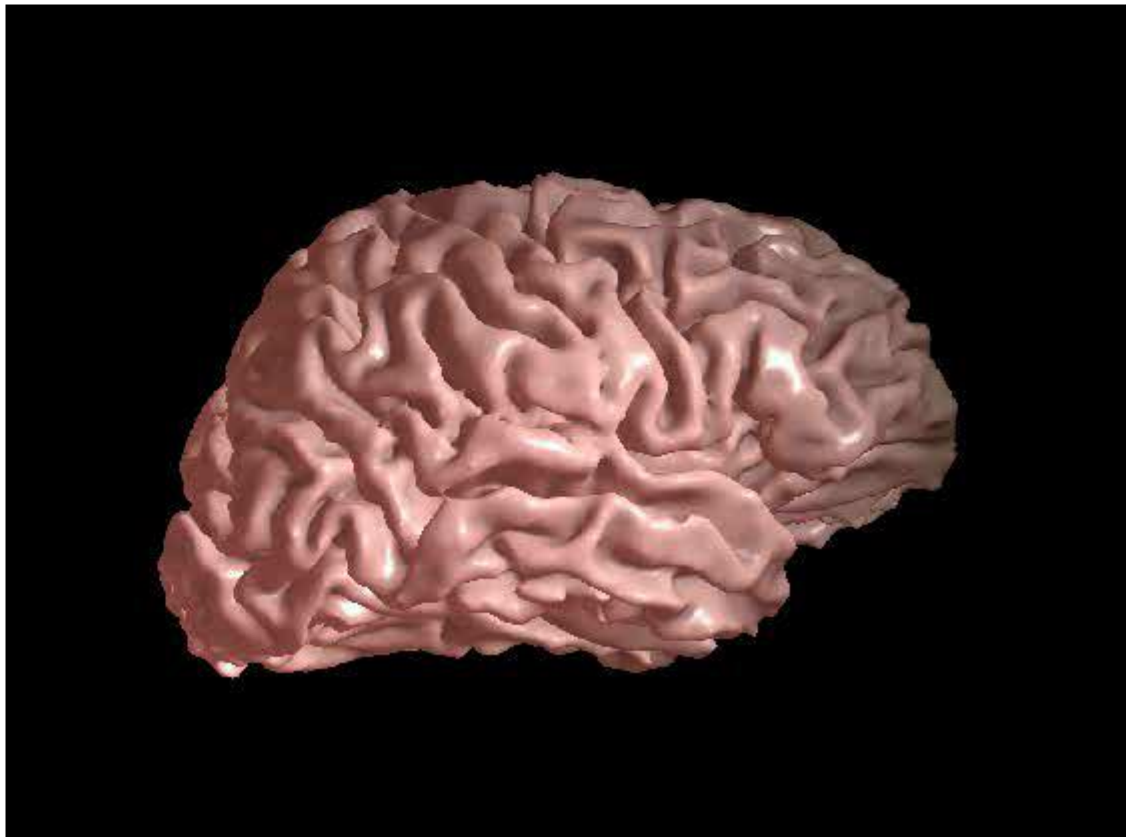
Example 5 A snapshot of room temperature T_i at time t_0 is an example of a spatial variable, i.e. $T_i = f(t_0, \xi, \eta, \zeta) = f(\xi, \eta, \zeta)$.

Example 6 Spatial variable (temperature at frozen time)

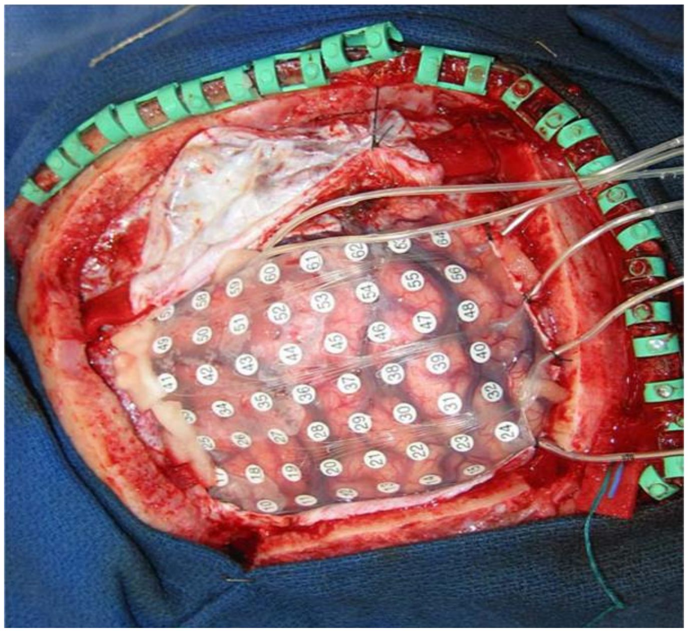


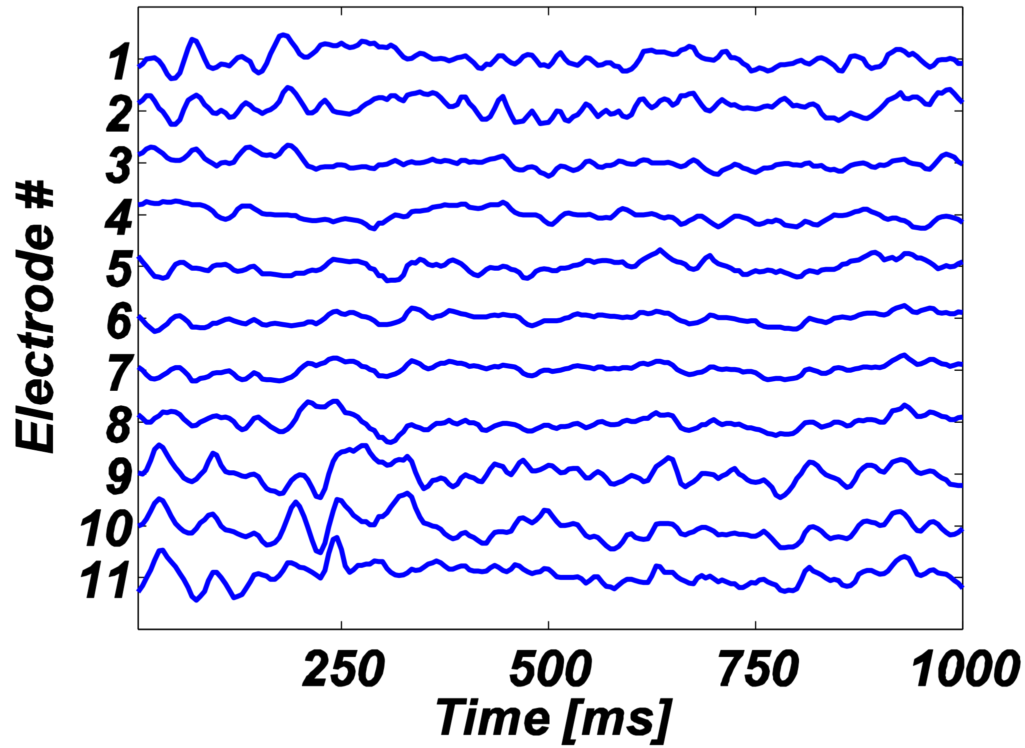
Biomedical Examples

Example 7 Spatio-temporal signal (movie of a spinning brain)



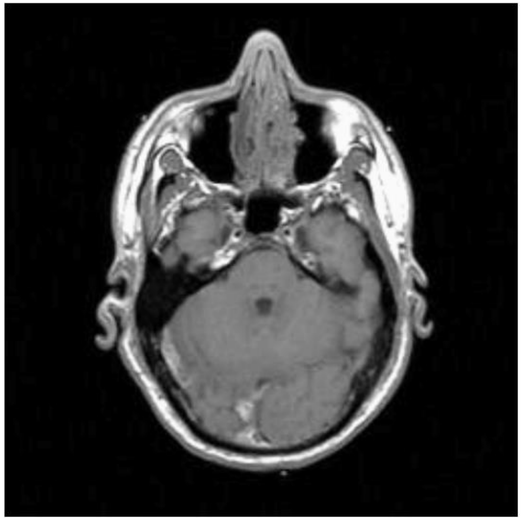
Example 8 Spatio-temporal signal (human electrocorticogram ECoG)



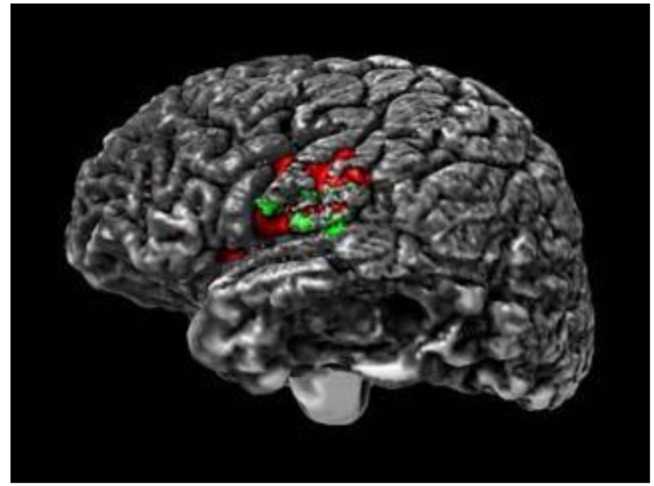


Example 9 Spatial signals (medical imaging)

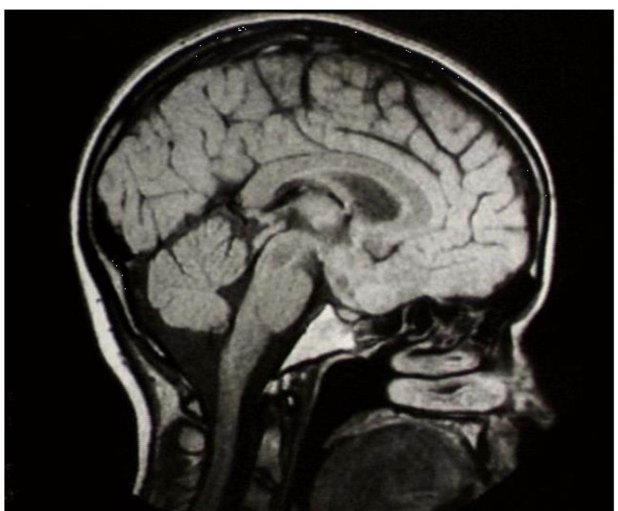
MRI



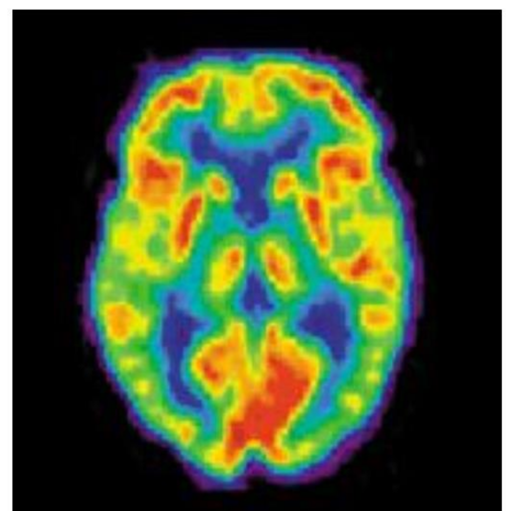
fMRI



CAT

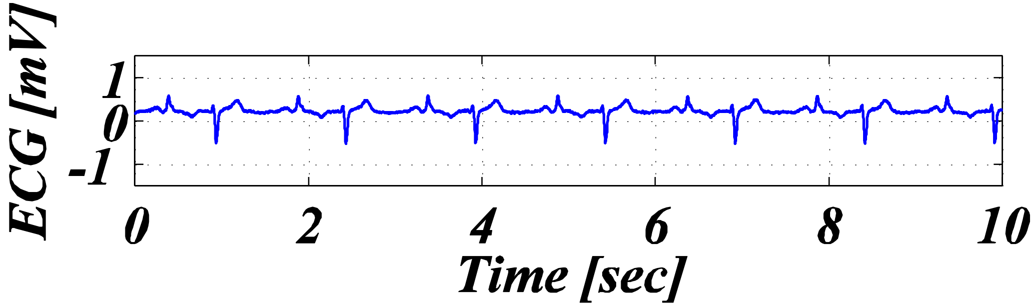


PET



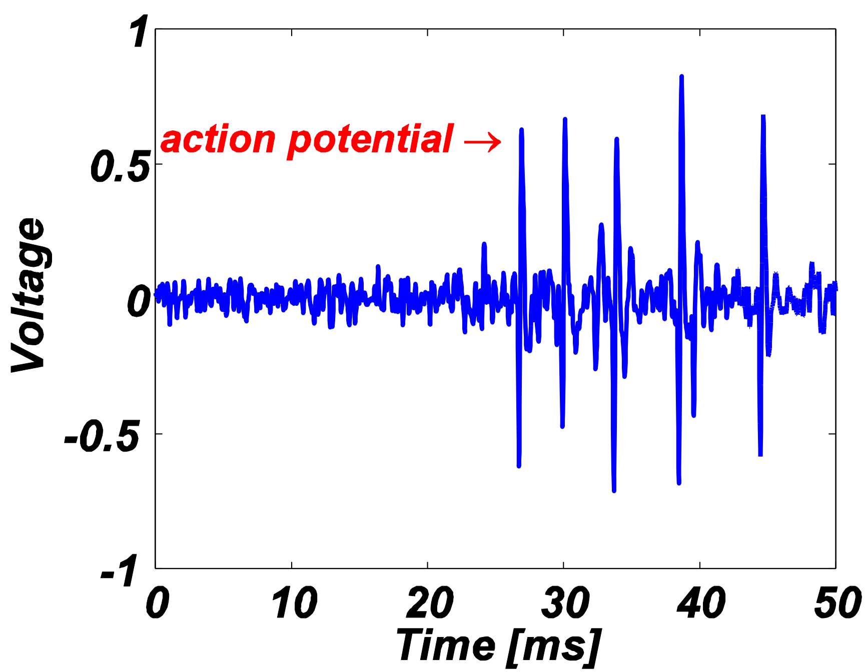
Example 10

Temporal signal (ECG)
Source: http://www.physionet.org/physiobank
(database of physiologic signals)



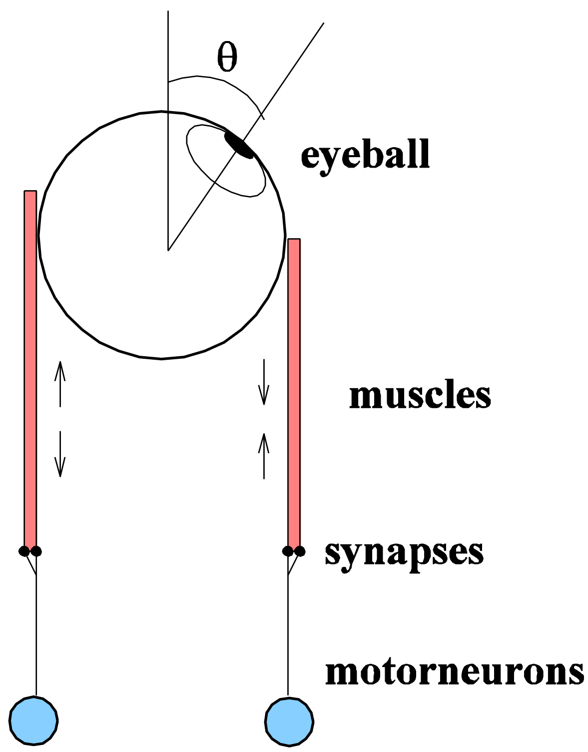
Example 11

Action potentials (from the monkey brain)



Mathematical Models

Eye movement system



Eye movement model

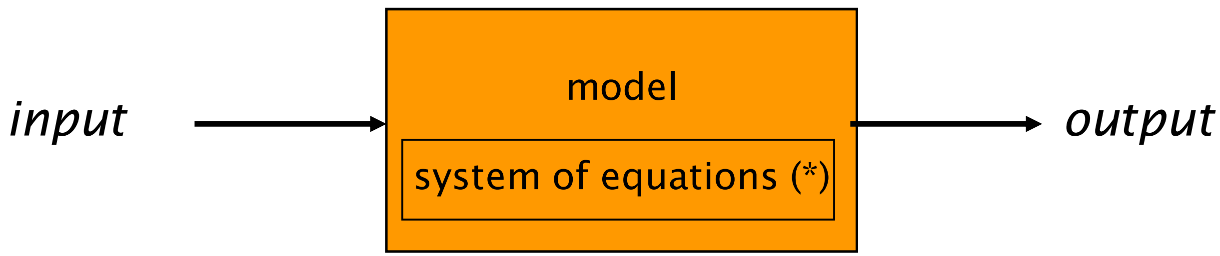
$$J\ddot{\theta}(t) + B\dot{\theta}(t) + K\,\underbrace{\theta(t)}_{output} = \underbrace{\tau(t)}_{input}$$

$$J, B, K \;\; - parameters$$

(demo: `spring_mass_simulation.m`)

Definition 1 *A formal description of a system using mathematical symbols, relations, operations, etc., is called the mathematical model of the system.*

Schematically



Note: We will often use terms system and model interchangeably.

Based on their characteristics, we classify models in several ways (underlined options are more realistic):

Deterministic (variables deterministic functions)
<u>Stochastic</u> (variables random functions)

Spatially $\begin{cases} \text{Concentrated ((*) is a (system of) ODEs)} \\ \underline{\text{Distributed}} \text{ ((*) is a (system of) PDEs)} \end{cases}$

Static ((*) contains no time derivatives)
<u>Dynamic</u> ((*) contains time derivatives)

Time-invariant (system parameters do not depend explicitly on *t*)
<u>Time-varying</u> (system parameters depend explicitly on *t*)

With time delay (variables time-lagged)
Without time delay (variables not time-lagged)

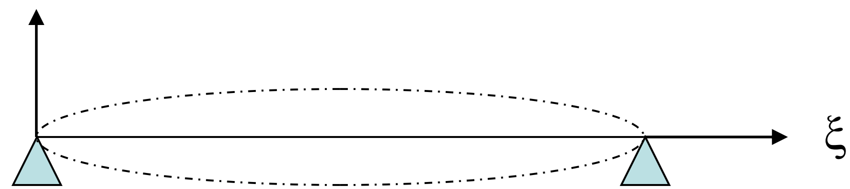
Linear ((*) depends on variables in a linear fashion)
<u>Nonlinear</u> ((*) depends on variables in a non-linear fashion)

<u>Time-continuous</u> (variables defined for continuous *t*)
Time-discrete (variables defined for discrete *t*)

Note: Time-varying systems are also known as nonstationary systems. Similarly, time-invariant systems are called stationary systems.

Example 12 1-D wave equation (equation of a vibrating string)

$y(t,\xi)$ – deflection of string

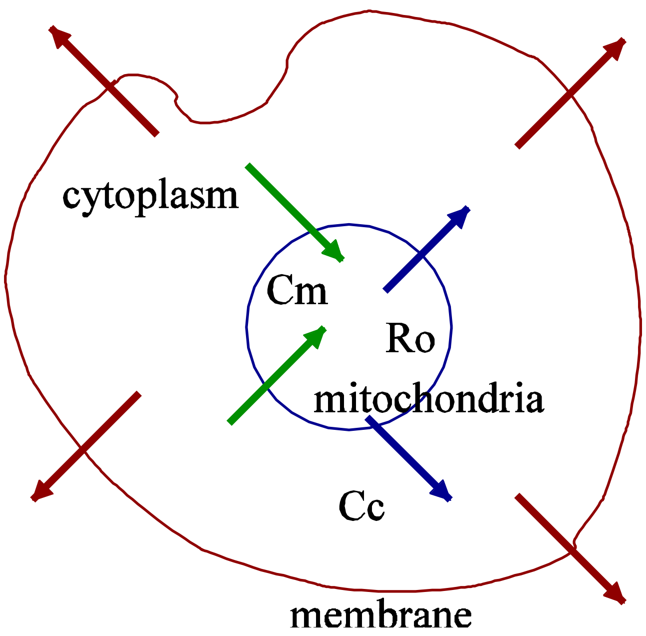


$$\frac{\partial^2 y(t,\xi)}{\partial t^2} = \theta^2 \frac{\partial^2 y(t,\xi)}{\partial \xi^2}$$

- deterministic
- spatially distributed,
- dynamic
- time-invariant
- without delay
- linear
- time-continuous

Note: differentiation is a linear operation

Example 13 Cellular dynamics



$$V_m \dot{C}_m(t) = R_0(t) - K_{12}[C_m(t) - C_c(t)]$$
$$V_c \dot{C}_c(t) = K_{12}[C_m(t) - C_c(t)] - K_2 C_c(t)$$

- deterministic
- spatially concentrated
- dynamic
- time-invariant
- without delay
- linear
- time-continuous

Example 14

Population dynamics

$$N(t+1) = (1+\lambda)N(t) \qquad (1)$$

- deterministic
- spatially concentrated
- dynamic
- time-invariant
- without delay
- linear
- time-discrete

Note: For time-discrete systems, derivatives are replaced by finite differences ((1) can be derived from $dN(t)/dt = rN(t)$)

$$N(t+1) = [1+\lambda(t)]N(t)$$

- deterministic
- spatially concentrated
- dynamic
- time-varying
- without delay
- linear
- time-discrete

Example 15 Gene regulatory networks (*E. coli*, Gardner et al. *Nature*, 2000)

$$\dot{u}(t) = -u(t) + \frac{\alpha_u}{1 + v(t)^{\beta}}$$
$$\dot{v}(t) = -v(t) + \frac{\alpha_v}{1 + u(t)^{\gamma}}$$

- deterministic
- spatially concentrated
- dynamic
- time-invariant
- without delay
- nonlinear
- time-continuous

Example 16 Cell division (logistic equation)

$$\dot{N}(t) = rN(t)\left(1 - \frac{N(t)}{K}\right)$$

$N(t)$ the number of cells at time t
r, K – parameters
r – growth rate due to division
K – carrying capacity

- deterministic
- spatially concentrated
- dynamic
- time-invariant
- without delay
- nonlinear
- time-continuous

Example 17 Model of the heart ventricle

$$V(t) = V_d + C(t)P(t)$$

$V(t)$ - volume
$P(t)$ - pressure
V_d, C – parameters
V_d – the dead volume ($V = V_d$ @ $P = 0$)
C - compliance

- deterministic
- spatially concentrated
- static
- time-varying
- without delay
- linear (affine)
- time-continuous

Introduce substitution: $\bar{V}(t) \coloneqq V(t) - V_d$

$$\bar{V}(t) = C(t)P(t)$$

- deterministic
- spatially concentrated
- static
- time-varying
- without delay
- linear
- time-continuous

Example 18 Model of anesthesia

$$\dot{x}(t) = -\frac{1}{\tau}x(t) + \frac{k}{\tau}u(t)$$
$$y(t) = x(t - L)$$

- deterministic
- spatially concentrated
- dynamic
- time-invariant
- with time delay
- linear
- time-continuous

y – change in the mean arterial pressure (output)
u – the infusion rate of a hypotensive drug (input)
τ, k, L – parameters
τ – time constant
k – gain factor
L – dead time (transport lag)

Problems to think about

Example 19 The diffusion of oxygen in a living tissue

$$\frac{\partial y(\xi,\eta,\zeta,\mathrm{t})}{\partial t} = D\left(\frac{\partial^2 y(\xi,\eta,\zeta,t)}{\partial \xi^2} + \frac{\partial^2 y(\xi,\eta,\zeta,t)}{\partial \eta^2} + \frac{\partial^2 y(\xi,\eta,\zeta,t)}{\partial \zeta^2}\right) - ky(\xi,\eta,\zeta,t)$$

- $y(\xi,\eta,\zeta,t) \in \mathbb{R}$ is the oxygen concentration at a point (ξ,η,ζ) and time t
- D is the diffusion constant
- k is the oxygen uptake constant

Example 20

Excitability of barnacle giant muscle fiber (Morris and Lecar, *Biophys. J.*, 1981)

$$
\begin{aligned}
C\dot{V}(t) &= I(t) - g_{Ca} m_\infty(V)(V(t) - V_{Ca}) \\
&\quad - g_K w(t)(V(t) - V_K) - g_L(V - V_L) \\
\tau_m(V)\,\dot{m}(t) &= \phi_m(m_\infty(V) - m(t)) \\
\tau_w(V)\,\dot{w}(t) &= \phi_w(w_\infty(V) - w(t))
\end{aligned}
$$

where

$$
\begin{aligned}
m_\infty(V) &= 0.5\left(1 + \tanh\left(\frac{V - V_1}{V_2}\right)\right) \\
\tau_m(V) &= \frac{1}{\cosh((V - V_1)/2V_2)} \\
w_\infty(V) &= 0.5\left(1 + \tanh\left(\frac{V - V_3}{V_4}\right)\right) \\
\tau_w(V) &= \frac{1}{\cosh((V - V_3)/2V_4)}
\end{aligned}
$$

Example 21 Predator-prey equations (Lotka-Volterra)

$$\dot{x}(t) = \alpha \mathrm{x}(t) - \beta x(t)y(t)$$
$$\dot{y}(t) = \delta x(t)y(t) - \gamma y(t)$$

x - the size of prey population at time t;
y - the size of predator population at time t;
$\alpha, \beta, \gamma, \delta$ - parameters representing the interaction of the two species.

Example 22 Dynamics of emotions (S. Strogatz, *Mathematics Magazine*, 1988)

$$\dot{R}(t) = -aJ(t)$$
$$\dot{J}(t) = bR(t)$$

$R(t)$ – Romeo's love $(R > 0)$/hate $(R < 0)$ for Juliet
$J(t)$ – Juliet's love $(J > 0)$/hate $(J < 0)$ for Romeo
$a, b > 0$

Example 23

Enzyme kinetics (Michaelis-Menten)

$$E + S \underset{k_{-1}}{\overset{k_1}{\rightleftarrows}} ES \overset{k_2}{\rightarrow} E + P$$

E - enzyme, S - substrate, ES - enzyme-substrate complex, P - product

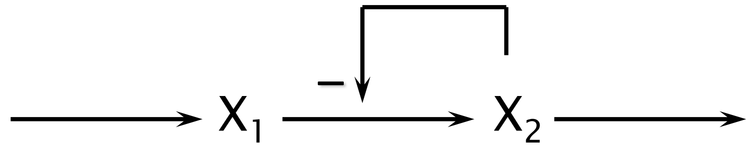
k_1, k_2, k_{-1} - the reaction rate constants

$[S], [ES], [P]$ - concentrations

$$\begin{aligned}
\frac{d[S]}{dt} &= -k_1[E][S] + k_{-1}[ES] \\
\frac{d[ES]}{dt} &= k_1[E][S] - k_{-1}[ES] - k_2[ES] \\
\frac{d[P]}{dt} &= k_2[ES]
\end{aligned}$$

Example 24

Biosynthetic pathway with feedback inhibition



$$\frac{d[X_1]}{dt} = 1 - [X_1]^2[X_2]^{-2}$$

$$\frac{d[X_2]}{dt} = [X_1]^2[X_2]^{-2} - [X_2]$$