Laplace Transform

Motivation: Recall a homogeneous differential equation, e.g.

$$\ddot{y}(t) + a_1\,\dot{y}(t) + a_0 y(t) = 0$$

To solve it, we form an algebraic equation: $\underbrace{s^2 + a_1 s + a_0}_{f(s)} = 0$

Note: Eq. $f(s) = 0$ is called the <u>characteristic equation</u> (CE) (of the differential equation above).

Note: The polynomial $f(s)$ is the <u>characteristic polynomial</u>.

Note: Finding the roots of CE is equivalent to finding the zeros of $f(s)$.

To get CE from a differential equation: $\dfrac{d^k y(t)}{dt^k} \leftrightarrow s^k \quad k = 0, 1, \cdots, n$

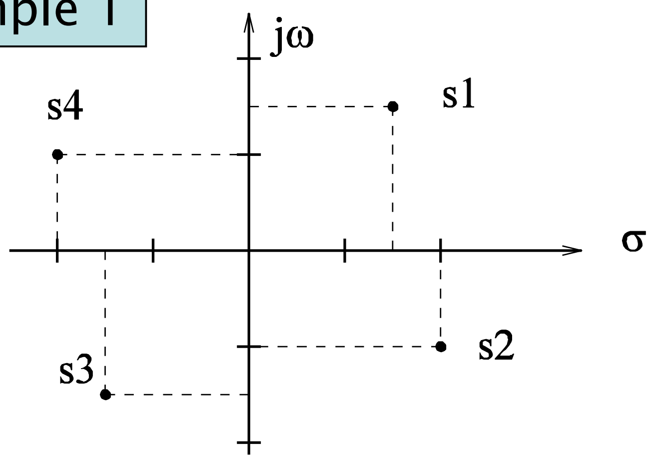
Where is this coming from? We'll see in a few slides.

Definition 1 *Let x be an arbitrary function of time. If the following integral exists:*

$$\boxed{X(s) = \int_0^\infty x(t)e^{-st}dt} \qquad (1)$$

where s is a complex number ($s = \sigma + j\omega$), it is called the Laplace transform of x.

Example 1



$$s_1 = 1.5 + 1.5j \quad s_2 = 2 - j$$
$$s_3 = -1.5 - 1.5j \quad s_4 = -2 + j$$

Note: $\sigma = \mathcal{R}e(s) \quad \omega = \mathcal{I}m(s)$

Laplace transform may not exist for an arbitrary $x(t)$ or an arbitrary s.

Definition 2 *If there exists a real number* γ *such that the following two integrals*

$$X^{-}(s) = \lim_{\substack{\alpha\to 0^{-}\\ \beta\to\infty}} \int_{\alpha}^{\beta} x(t)e^{-st}\,dt$$

and

$$X^{+}(s) = \lim_{\substack{\alpha\to 0^{+}\\ \beta\to\infty}} \int_{\alpha}^{\beta} x(t)e^{-st}\,dt$$

converge for all complex numbers s *with* $\mathcal{R}e(s) > \gamma$, *then* $X^{-}(s)$ *and* $X^{+}(s)$ *are called the left and right Laplace transforms of* x, *respectively. The smallest number* γ *that satisfies the above condition is called the abscissa of absolute convergence.*

Theorem 1 *The Laplace transform (1) of a function* x *exists if and only if* $X^{-}(s)$ *and* $X^{+}(s)$ *exist and they are equal.*

The Laplace transform $X(s)$ of a function x is often denoted by: $\mathcal{L}\{x(t)\} = X(s)$

It can be shown that if a function $x(t)$ satisfies several conditions, then its Laplace transform $X(s)$ exists.

We will not list all the conditions here. One of the basic requirements is that:

$$x(t) = 0, \qquad t < 0$$

That's why the Laplace transform integral (1) starts from 0.

In particular, this Laplace transform is called <u>one-sided</u> or <u>unilateral</u>. It is designed for the treatment of causal signals (signals that are 0 for negative t).

It suffices to know that all the functions we are going to use in the context of LTI systems will satisfy these conditions.

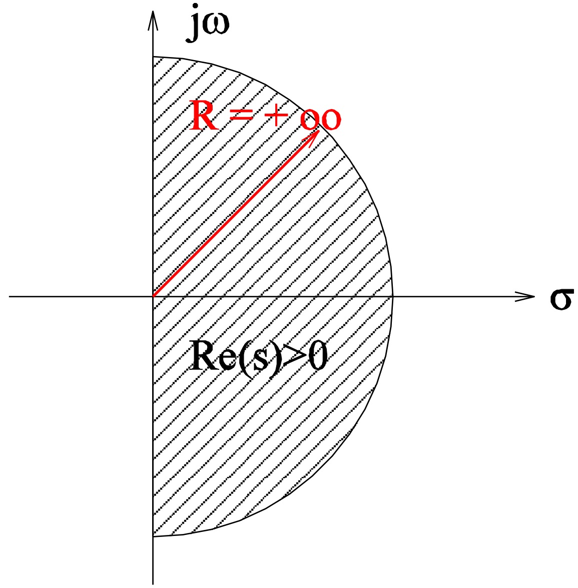
Example 1 Find $\mathcal{L}\{x(t)\}$ where $x(t) = h(t)$ (Heaviside function). Since $h(t) = 0$ for $t < 0$, we have $X^-(s) = X^+(s)$, thus $\mathcal{L}\{x(t)\}$ exists. Therefore:*

$$X(s) = \frac{1}{s}$$

The abscissa of absolute convergence: $\gamma = 0$. Why is this true?

$$\begin{aligned}
\lim_{\beta\to\infty} e^{-s\,\beta} &= \lim_{\beta\to\infty} e^{-(\sigma+j\omega)\,\beta} \\
&= \lim_{\beta\to\infty} \left[e^{-\sigma\,\beta} \quad \underbrace{e^{-j\omega\,\beta}}_{\cos(\omega\beta)-j\sin(\omega\beta)} \right] \\
&= \lim_{\beta\to\infty} \underbrace{e^{-\sigma\,\beta}}_{\downarrow} \underbrace{[\cos(\omega\beta) - j\sin(\omega\beta)]}_{\text{finite number } \forall\beta} \\
&= \text{will not converge if } \sigma \leq 0
\end{aligned}$$

Since $\sigma = \mathcal{R}e(s)$, for convergence we need: $\mathcal{R}e(s) > 0$.



Shaded area = right half-plane (RHP).

Example 2

Find $\mathcal{L}\{x(t)\}$ where $x(t) = \delta(t)$ (Dirac function). Claim:

$$X^{-}(s) = 1 \qquad X^{+}(s) = 0$$

Properties

1. Linearity property: $\mathcal{L}\left\{\sum_{k=1}^{n} \alpha_k x_k(t)\right\} = \sum_{k=1}^{n} \alpha_k \mathcal{L}\{x_k(t)\}$

Proof: follows directly from the definition.*

2. Time-differentiation rule: $\mathcal{L}\{\dot{x}(t)\} = sX(s) - x(0)$, where $x(t)$ is a differentiable function?

Proof:

$$\begin{aligned}
\mathcal{L}\{\dot{x}(t)\} &= \int_0^\infty \dot{x}(t) e^{-st}\, dt \\
&\quad \left| \begin{array}{rlrl} U &= e^{-st} & dV &= \dot{x}(t)dt \\ dU &= -se^{-st} & V &= x(t) \end{array} \right| \\
\mathcal{L}\{\dot{x}(t)\} &= [UV]_0^\infty - \int_0^\infty V\, dU = \underbrace{\left[x(t)e^{-st}\right]_0^\infty}_{\mathcal{R}e(s)>0} + s \underbrace{\int_0^\infty x(t)e^{-st}\, dt}_{\mathcal{L}\{x(t)\}} \\
&= [0 - x(0)] + sX(s) = sX(s) - x(0)
\end{aligned}$$

Similarly, we can show that for any twice differentiable function *x(t)* we have:

$$\mathcal{L}\{\ddot{x}(t)\} = s \underbrace{\mathcal{L}\{\dot{x}(t)\}}_{sX(s)-x(0)} - \dot{x}(0) = s^2 X(s) - s x(0) - \dot{x}(0)$$

2. Time-differentiation rule (in general)

$$\mathcal{L}\left\{\frac{d^k x(t)}{dt^k}\right\} = s^k X(s) - \sum_{i=1}^{k} s^{i-1} x^{(k-i)}(0)$$

Proof: by induction.

But how does this help?

Example 3 $\qquad \ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) = 0$

Take the Laplace transform of the both sides (assume 0 initial conditions) to get* $\quad s^2 + a_1 s + a_0 = 0$ -CE of the diff. eq. above

3. Time-integration rule: $\mathcal{L}\left\{\int_0^t x(\tau)\,d\tau\right\} = \frac{1}{s}X(s)$

where x is an integrable function satisfying $\int_{0^-}^{0^+} x(\tau)\,d\tau = 0$

Proof*

4. Time-scaling property: $\mathcal{L}\left\{x\left(\frac{t}{\alpha}\right)\right\} = \alpha X(\alpha s)$

where $\alpha > 0$.

Proof (easy)*

5. Time-delay property: $\mathcal{L}\left\{x(t-T)\right\} = e^{-Ts}X(s)$

Proof (easy)*

6. Initial Value Theorem: $\lim_{t \to 0+} x(t) = \lim_{s \to \infty} sX^+(s)$

Note: If $X(s)$ exists, it must be equal to $X^+(s)$, therefore

$$\lim_{t \to 0+} x(t) = \lim_{s \to \infty} sX(s)$$

Proof: recall $\mathcal{L}\{\dot{x}(t)\} = sX(s) - x(0)$

therefore $$\mathcal{L}\{\dot{x}(t)\} = sX^+(s) - \underbrace{x(0^+)}_{\lim_{t \to 0+} x(t)}$$

$$\lim_{s \to \infty} \mathcal{L}^+\{\dot{x}(t)\} = \lim_{s \to \infty} sX^+(s) - x(0^+)$$

On the other hand: $\lim_{s \to \infty} \mathcal{L}^+\{\dot{x}(t)\} = \lim_{s \to \infty} \int_{0+}^{\infty} \dot{x}(t) e^{-st}\, dt = 0$

From the last two equations: $x(0^+) = \lim_{s \to \infty} sX^+(s)$

7. Final Value Theorem: $\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX^{+}(s)$

Note: If $X(s)$ exists, it must be equal to $X^{+}(s)$, therefore

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$$

Proof:

$$\begin{aligned}
\mathcal{L}^{+}\{\dot{x}(t)\} &= sX^{+}(s) - x(0^{+}) \\
\lim_{s \to 0} \mathcal{L}^{+}\{\dot{x}(t)\} &= \lim_{s \to 0} sX^{+}(s) - x(0^{+})
\end{aligned}$$

On the other hand:

$$\begin{aligned}
\lim_{s \to 0} \mathcal{L}^{+}\{\dot{x}(t)\} &= \lim_{s \to 0} \int_{0+}^{\infty} \dot{x}(t) e^{-st}\, dt = \int_{0+}^{\infty} \dot{x}(t)\, dt \\
&= \int_{0+}^{\infty} dx(t) = \underbrace{x(\infty)}_{\lim_{t \to \infty} x(t)} - x(0^{+})
\end{aligned}$$

From the last two equations: $\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX^{+}(s)$

8. Frequency-shift: $\mathcal{L}\left\{e^{s_0 t}x(t)\right\} = X(s - s_0)$

Proof (easy)*

The Inverse Laplace Transform

Function	$\longrightarrow$	Laplace Transform
$x(t)$	$\longrightarrow$	$X(s)$
Time domain	$\longrightarrow$	Complex domain
t	$\longrightarrow$	s

Can we go back: $X(s) \rightarrow x(t)$?

The inverse Laplace Transform: if a function x is continuous at some time t, then:

$$x(t) = \underbrace{\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s)e^{st}\,ds}_{\mathcal{L}^{-1}\{X(s)\}} \qquad c \in (\gamma, \infty)$$

where γ is the abscissa of absolute convergence.

Fortunately, we will not calculate the inverse Laplace Transform, which requires the knowledge of complex analysis.

Reason: the inverse Laplace Transforms used in the theory of LTI systems have been calculated and tabulated.

In fact, the method relies on calculating $X(s)$ from $x(t)$ and using the fact that

$$x(t) = \mathcal{L}^{-1}\{X(s)\}$$

Example 4 Calculate the inverse Laplace Transform of $X(s) = 1/s$.

We already calculated $\mathcal{L}\{h(t)\} = \frac{1}{s}$, therefore: $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = h(t)$.

Example 5 $x(t) = e^{\alpha t} h(t)$

By the property 8, we have $X(s) = H(s - \alpha)$, where $H(s) = \mathcal{L}\{h(t)\} = 1/s$.

Therefore: $\mathcal{L}\{e^{\alpha t} h(t)\} = 1/(s - \alpha)$. Consequently: $\mathcal{L}^{-1}\left\{\frac{1}{s - \alpha}\right\} = e^{\alpha t} h(t)$

Example 6 $x(t) = \sin(\omega t) h(t)$

Answer:* $\mathcal{L}\{\sin(\omega t) h(t)\} = \dfrac{\omega}{s^2 + \omega^2}$

Consequently: $\mathcal{L}^{-1}\left\{\dfrac{\omega}{s^2 + \omega^2}\right\} = \sin(\omega t) h(t)$

Example 7 $x(t) = \cos(\omega t)h(t)$

Answer:* $\mathcal{L}\{\cos(\omega t)h(t)\} = \dfrac{s}{s^2+\omega^2}$

Consequently: $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+\omega^2}\right\} = \cos(\omega t)h(t)$

Therefore, all the major functions and their Laplace transforms have been tabulated.

With the help of the table, it is easy to switch from the time domain to the complex domain, and vice versa.

Time Domain		Complex domain
$x(t)$	$\mathcal{L} \longrightarrow$	$X(s)$
$x(t)$	$\longleftarrow \mathcal{L}^{-1}$	$X(s)$

Time function	Laplace Transform
$x(t)$, $t \geq 0$	$X(s)$
$\delta(t)$	$X^{+}(s) = 0$ $X^{-}(s) = 1$ $X(s)$ - does not exist
$h(t)$	$\frac{1}{s}$
$t\,h(t)$	$\frac{1}{s^2}$
$t^{n-1}\,h(t)$	$\frac{(n-1)!}{s^n}$
$e^{\alpha\,t}\,h(t)$	$\frac{1}{s-\alpha}$
$t^{n-1}e^{\alpha\,t}\,h(t)$	$\frac{(n-1)!}{(s-\alpha)^n}$
$\left(1 - e^{-\alpha t}\right) h(t)$	$\frac{\alpha}{s(s+\alpha)}$
$e^{-\alpha\,t}\cos(\omega t) h(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$
$\vdots$	$\vdots$

So, what is this good for?

Example 8 Solve $\dot{y}(t) + y(t) = 1 \quad t \geq 0$

with the initial condition $y(0) = -2$ using the Laplace Transform

Answer: $y(t) = (1 - 3e^{-t})h(t)$

If the Inverse Laplace Transform cannot be found in a table, there are other techniques that will work.

Partial fraction expansion (Heaviside) is one of them (past).

Symbolic software (e.g. MAPLE) is much faster.