

# Frequency Response

Recall that the transfer function  $T(s)$  links the input and the output through:

$$T(s) = Y(s)/U(s)$$

under 0 initial conditions.

Also recall that (in general)  $s = \sigma + j\omega$ .

Because we are interested in LTI systems, we may want to restrict ourselves to  $s = j\omega$ . In other words we may be interested in the way the system behaves at different frequencies.

Why?

Fourier series expansion: many functions can be expressed as a linear combination of sines and cosines:

$$u(t) = \sum_{i=0}^{\infty} a_i \cos(\omega_i t) + \sum_{i=1}^{\infty} b_i \sin(\omega_i t)$$

Therefore, if we know how an LTI system treats individual frequencies  $\omega_i$   $i = \{1, 2, \dots\}$ , we will know how the system treats any input:

$$L[t; x(0), u_{[0,t]}] = \sum_{i=0}^{\infty} a_i \underbrace{L[t; x(0), c_{[0,t]}^{(i)}]}_{\text{response to } \cos(\omega_i t)} + \sum_{i=1}^{\infty} b_i \underbrace{L[t; x(0), s_{[0,t]}^{(i)}]}_{\text{response to } \sin(\omega_i t)}$$

where  $c_{[0,t]}^{(i)}$  and  $s_{[0,t]}^{(i)}$  are  $\cos(\omega_i t)$  and  $\sin(\omega_i t)$ , respectively.

Thus, to obtain the response of an LTI system to an arbitrary input  $u(t)$ , it suffices to know its response to sines and cosines at frequencies  $\omega = \omega_i$ .

It turns out that the response of an LTI system to a periodic input  $\sin(\omega_i t)$  or  $\cos(\omega_i t)$ , is easy to calculate.

It suffices to evaluate the transfer function  $T(s)$  at points  $s = j\omega_i$ .

**Definition 1** Let  $T(s)$  be a transfer function of an LTI system. The frequency response of the system, denoted by  $F(j\omega)$ , is defined as

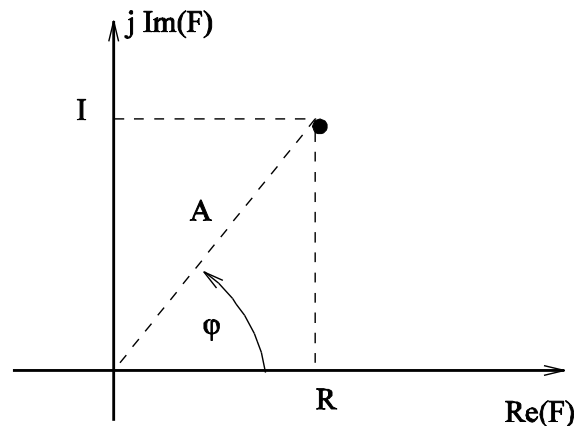
$$F(j\omega) := T(s)|_{s=j\omega}$$

♦  $F(j\omega)$  is a complex number.

We can write  $F(j\omega) = R(\omega) + jI(\omega)$ , where  $R(\omega) = \mathcal{R}_e\{F(j\omega)\}$  and  $I(\omega) = \mathcal{I}_m\{F(j\omega)\}$ . More conveniently, we can use the polar form:

$$F(j\omega) = \underbrace{A(\omega)}_{\text{amplitude}} e^{j \overbrace{\varphi(\omega)}^{\text{phase}}}$$

where  $A(\omega) = |F(j\omega)|$  is the modulus of  $F(j\omega)$ , and  $\varphi(\omega) = \angle F(j\omega)$  is the phase.



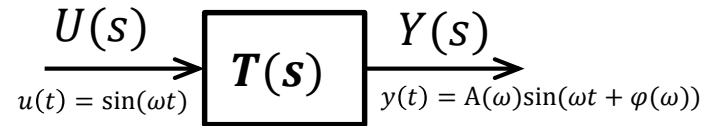
- ◆  $A(\omega)e^{j\varphi(\omega)} = A(\omega)[\cos(\varphi(\omega)) + j \sin(\varphi(\omega))]$
- ◆  $R(\omega) = A(\omega) \cos(\varphi(\omega)), \quad I(\omega) = A(\omega)\sin(\varphi(\omega))$
- ◆  $A(\omega) = \sqrt{R^2(\omega) + I^2(\omega)}$

Unlike  $A(\omega)$  there's no unique formula for  $\varphi(\omega)$ .

MATLAB command `atan2`

```
>> im = 1;  
>> re = -1;  
>> atan2(im,re)  
ans =  
2.3562  
>> ans*180/pi  
ans =  
135
```

Frequency response  $F(j\omega) = A(\omega)e^{j\varphi(\omega)}$  has a clear physical interpretation:



If a stable LTI system is excited by a **pure sine**, its steady-state response is going to be a **sine** of the **same frequency**, with the amplitude  $A(\omega)$  and the phase shift  $\varphi(\omega)$ .

◆  $A(\omega)$  is a frequency-dependent gain (output amplitude/input amplitude)

It seems natural to plot  $A(\omega)$  and  $\varphi(\omega)$  as a function of  $\omega$ .

Then we could just look at the plot and we would know the behavior of the system.

This is exactly the idea behind the Bode plot, except that the plots are against  $\log_{10} \omega$ , and instead of  $A(\omega)$  we plot  $L(\omega) = 20 \log_{10} A(\omega)$ .

Why do we use logarithms?

## Crash course in complex variables

For two complex numbers  $z_1$  and  $z_2$ , we have:

$$\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$$

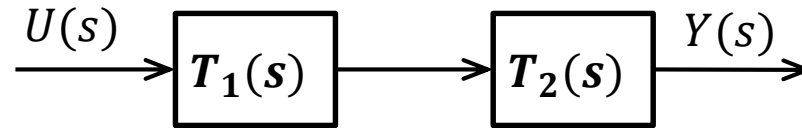
Likewise:  $\angle \frac{z_1}{z_2} = \angle z_1 - \angle z_2$

Similarly:  $|z_1 z_2| = |z_1| |z_2|$

Finally:  $\angle z_1 z_2 = \angle z_1 + \angle z_2$

### Example 7

Serial system:



We know that:  $T(s) = T_1(s)T_2(s)$ , therefore  $F(j\omega) = F_1(j\omega)F_2(j\omega)$

$$A(\omega) = |F(j\omega)| = |F_1(j\omega)F_2(j\omega)| = |F_1(j\omega)||F_2(j\omega)| = A_1(\omega)A_2(\omega)$$

$$\varphi(\omega) = \angle F(j\omega) = \angle F_1(j\omega)F_2(j\omega) = \angle F_1(j\omega) + \angle F_2(j\omega) = \varphi_1(\omega) + \varphi_2(\omega)$$

If we use a logarithmic amplitude:

$$L(\omega) \triangleq 20 \log_{10}(A(\omega)) = 20 \log_{10}(A_1(\omega)) + 20 \log_{10}(A_2(\omega)) = L_1(\omega) + L_2(\omega)$$

Thus we have addition instead of multiplication (if we had the plots of  $L_1$  and  $L_2$  we could just add them up).

## Example 2

## Canonical 1<sup>st</sup> order system

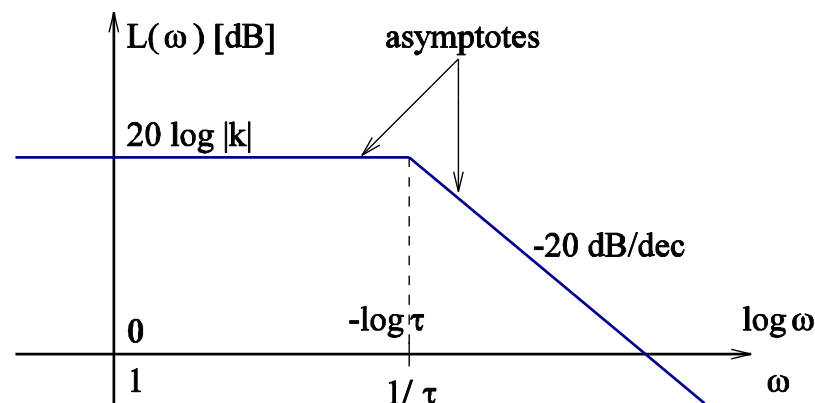
$$T(s) = \frac{k}{\tau s + 1} \Rightarrow F(j\omega) = \frac{k}{j\tau\omega + 1}$$

$$A(\omega) = \left| \frac{k}{j\tau\omega + 1} \right| = \frac{|k/\tau|}{|j\omega + 1/\tau|} = \frac{|k/\tau|}{\sqrt{\omega^2 + (1/\tau)^2}}$$

$$L(\omega) = 20 \log_{10} |k/\tau| - 20 \log_{10} \sqrt{\omega^2 + (1/\tau)^2}$$

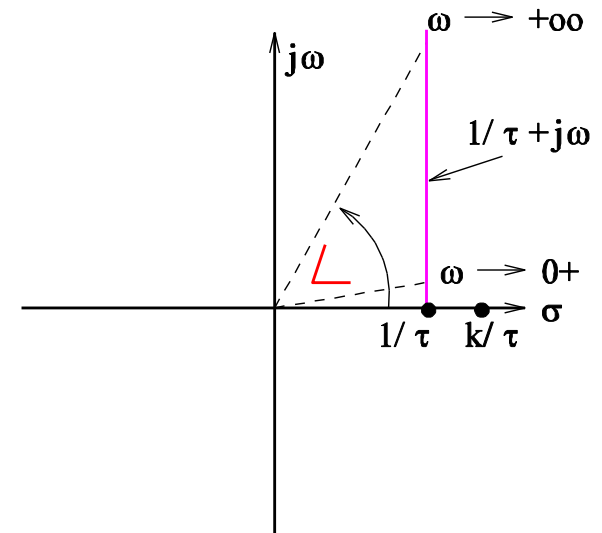
$$L(\omega) = 20 \log_{10} |k/\tau| - \begin{cases} 20 \log_{10} |1/\tau| & \omega \ll 1/\tau \\ 20 \log_{10} |\omega| & \omega \gg 1/\tau \end{cases}$$

$$L(\omega) = \begin{cases} 20 \log_{10} |k| & \omega \ll 1/\tau \\ \underbrace{20 \log_{10} |k/\tau| - 20 \log_{10} |\omega|}_{\text{linear function in } \log_{10} |\omega|} & \omega \gg 1/\tau \end{cases}$$



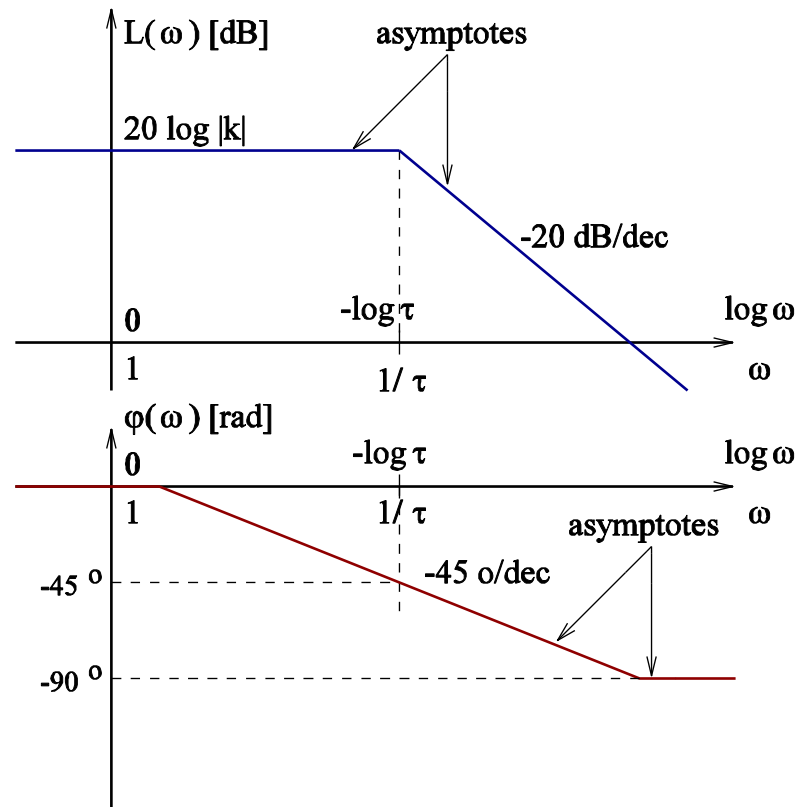
$$\begin{aligned}\varphi(\omega) &= \angle \frac{k}{j\tau\omega + 1} \\ &= \angle \frac{k/\tau}{j\omega + 1/\tau} = \underbrace{\angle k/\tau}_0 - \angle(j\omega + 1/\tau)\end{aligned}$$

$$\angle\left(j\omega + \frac{1}{\tau}\right) = \begin{cases} 0, & \omega \rightarrow 0^+ \\ ?, & \omega = 1/\tau \\ \pi/2, & \omega \rightarrow +\infty \end{cases}$$



Therefore:

$$\varphi(\omega) = \begin{cases} 0, & \omega \ll 1/\tau \\ -\pi/4, & \omega = 1/\tau \\ -\pi/2, & \omega \gg 1/\tau \end{cases}$$



Since the system

$$T(s) = \frac{k}{\tau s + 1}$$

attenuates the high frequencies ( $\omega \gg 1/\tau$ ), and approximately equally amplifies low frequencies ( $\omega \ll 1/\tau$ ), we say that the canonical first order LTI system acts like a low-pass filter.

MATLAB

```
>> n = 5;
```

```
>> d = [10 1];
```

```
>> tf(n,d)
```

Transfer function:

5

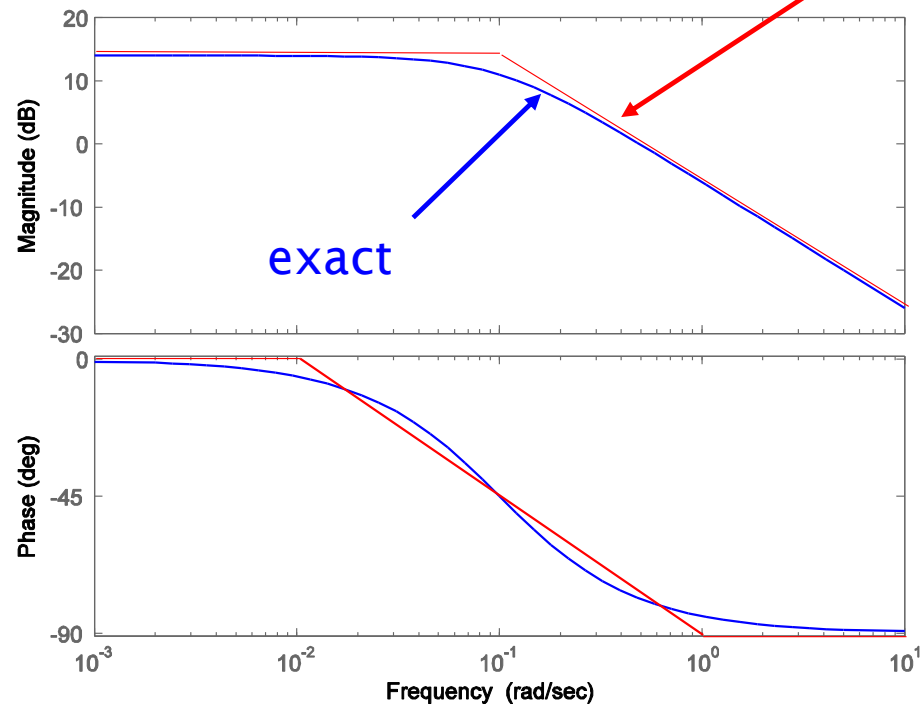
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10 s + 1

```
>> bode(n,d)
```

dc gain = 5

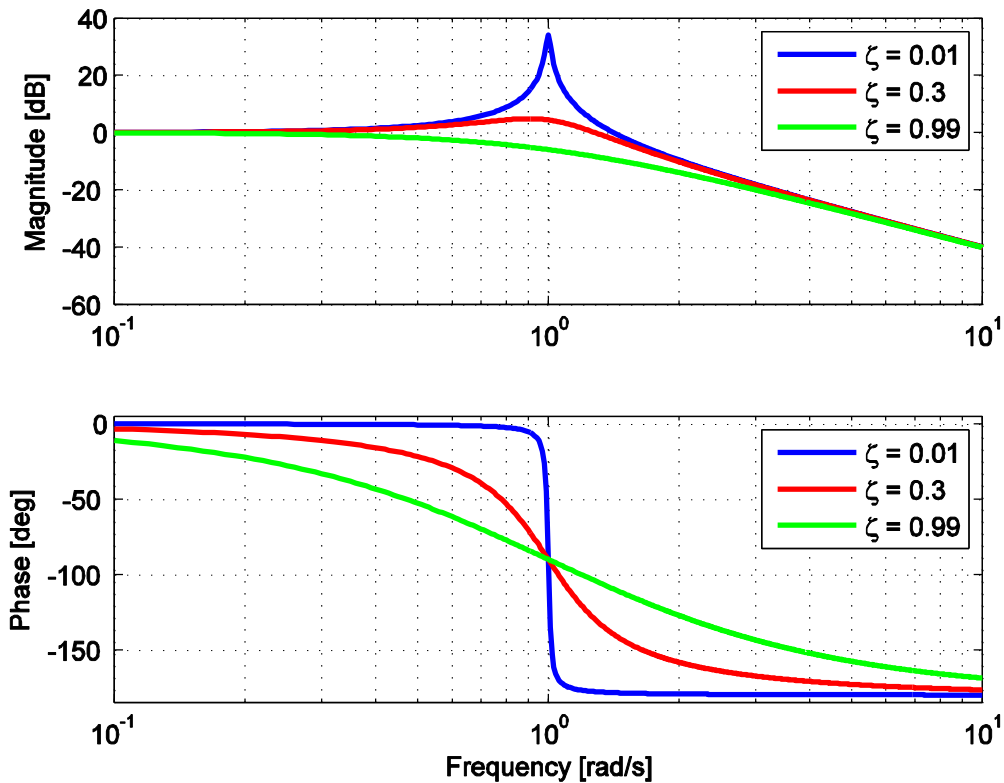
Bode Diagram



### Example 3

### Canonical 2<sup>nd</sup> order system

$$T(s) = \frac{k\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



$$k = 1$$

$$\omega_n = 1$$

$$\text{dc gain} = 1$$