Equilibrium States and Phase Portraits

Recall the definition of equilibrium x_e

The state (of a dynamic system) x_e is called the equilibrium state if and only if:

$$x(t) = x_e \quad \forall t \in [t^*, \infty)$$

and under <u>no input</u> conditions.

In other words, once the system reaches the state x_e at time t^*, the system never leaves it (unless input is applied).

Consequence:

$$x(t) = x_e \quad \forall t \in [t^*, \infty)$$

$$\text{or}$$

$$\dot{x}_e = 0$$

For LTI systems: $x_e = 0$ is always an equilibrium point.*

The number of equilibria in LTI systems is dictated by the state space matrix A.*

What about nonlinear systems? $\frac{dx(t)}{dt} = f(x(t), u(t), t)$

Time-invariance: $\frac{dx(t)}{dt} = f\big(x(t), u(t)\big)$ (no explicit dependence on t)

For equilibria (no inputs): $\frac{dx(t)}{dt} = f\big(x(t)\big)$

To find x_e , set $\frac{dx(t)}{dt} = 0$ or $f(x_e) = 0$ and solve for x_e.

It is not uncommon for nonlinear systems to have multiple equilibria.

Example 1: Logistic equation (population growth)

$$\frac{dN(t)}{dt} = rN(t)\left[1 - \frac{N(t)}{K}\right]$$

r – growth rate
K – carrying capacity

Example 2: Pendulum

$$mL\ddot{\theta}(t) + bL\dot{\theta}(t) + mg\sin\big(\theta(t)\big) = F(t)$$

Phase Portrait

Dynamical system: $\frac{dx(t)}{dt} = f(x(t), u(t), t)$

Time-invariant: $\frac{dx(t)}{dt} = f\big(x(t), u(t)\big)$ (no explicit dependence on t)

For equilibrium ($u(t) = 0$): $\frac{dx(t)}{dt} = f\big(x(t)\big)$ $(*)$

Initial condition: $x(0) = x_0$.

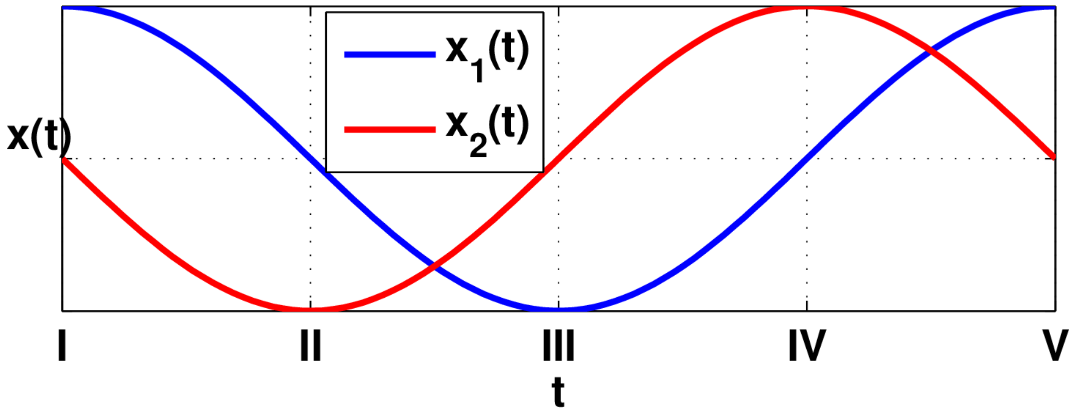
A function $x(t)$ that solves the differential equation above $(*)$, while satisfying the initial condition is called the solution.

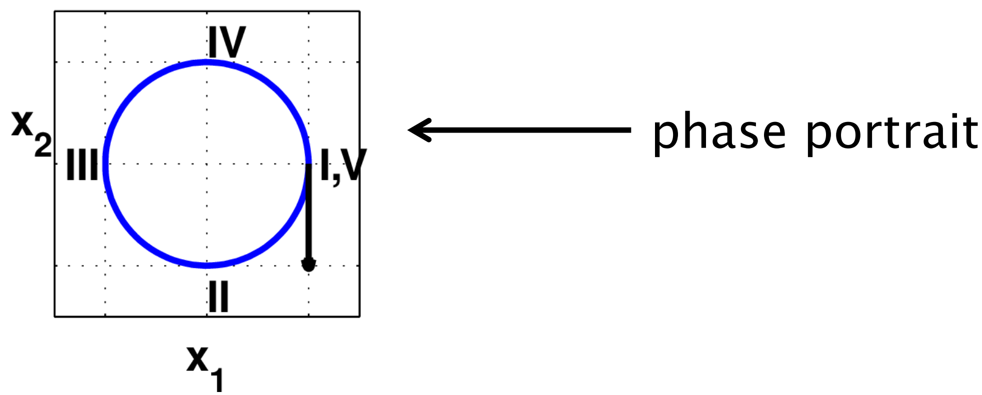
Collection of all solutions to (*) plotted as curves in the state space is called the phase portrait of (*). Phase portraits typically refer to 2-D (second order) systems

Note: Phase portrait is not a plot of $x(t)$ vs. t.

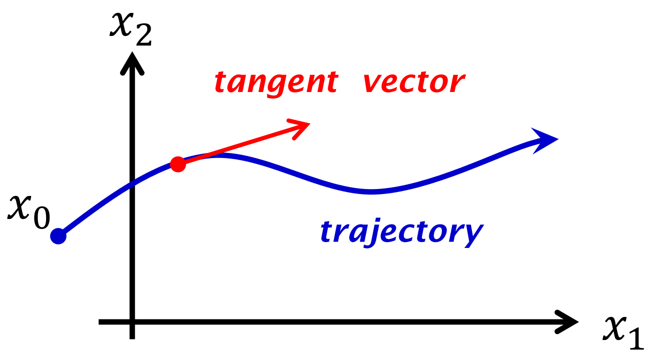
Example 3: Harmonic Oscillator:

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -x_1(t) \quad x(0) = [1, 0];$$





Time is lost in the phase portrait, but a lot is gained—geometry. Whether plotted as a function of time, or as a phase portrait, $x(t)$ is called the trajectory of a system.



Note: velocity (at any point) is the tangent vector to trajectory.

$$Velocity = \frac{dx(t)}{dt}$$

On the other hand: $\frac{dx(t)}{dt} = f\big(x(t)\big)$

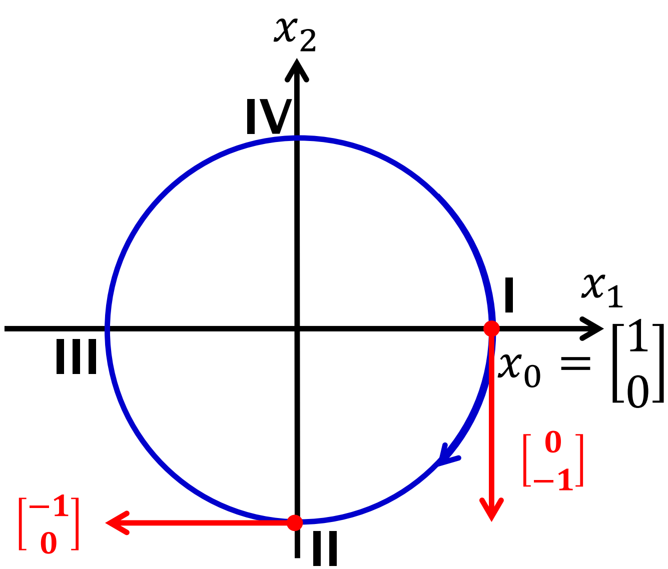
velocity vector vector field

For any point $(x_1(t), x_2(t))$ on the trajectory, $f(x(t))$ is a 2-D vector.

In mathematics, this is called a vector field.

Example 4: Harmonic Oscillator $\frac{dx(t)}{dt} = \underbrace{A\,x(t)}_{\textbf{vector field}}$ $\quad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x(t) = \underbrace{\begin{bmatrix} x_2(t) \\ -x_1(t) \end{bmatrix}}_{\textbf{vector field}}$$



$$\dot{x}(I) = \begin{bmatrix} 0 \\ -1 \end{bmatrix},\ \dot{x}(II) = \begin{bmatrix} -1 \\ 0 \end{bmatrix},\ \dot{x}(III) = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ \dot{x}(IV) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Since $f(x(t))$ is parameterized by t, such a vector field is often called the <u>flow of the system</u>

What about phase portraits, vector fields and flows of 1-D systems?

$$\frac{dx(t)}{dt} = f(x(t))$$

scalar scalar

Technically, $f(x(t))$ is not a vector field. Still the same formalism is useful.

Example 5: Logistic Equation*

$$\frac{dN(t)}{dt} = \underbrace{r\, N(t)\left[1 - \frac{N(t)}{K}\right]}_{f(N(t))}$$

Phase Portraits of 2-D LTI Systems

$$\dot{x}(t) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} x(t) \qquad x(0) = x_0$$

Solution: $\boxed{x(t) = e^{At}x_0}$ $\qquad e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$

$x(t)$ – depends on the eigenvalues of A!

$$\det(sI - A) = 0$$

$$s_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$$

where $\tau = a + d = \mathrm{trace}(A)$ and $\Delta = ad - bc = \det(A)$

Case I   Complex-Conjugate Poles ($s_{1,2} \in \mathbb{C}^{1\times 1}$)

- **stable focus (spiral)**
- **unstable focus (spiral)**
- center (marginally stable)

Case II   Real Poles ($s_{1,2} \in \mathbb{R}^{1\times 1}$)

case i	$0 > s_1 > s_2$		(**stable node**)
case ii	$s_1 > s_2 > 0$		(**unstable node**)
case iii	$s_1 > 0 > s_2$		(**saddle point**)
case iv	$s_1 > s_2 = 0$		(unstable line)
case v	$0 = s_1 > s_2$		(stable line)
case vi	$s_1 = s_2 > 0$	(2 lin. ind. eigenvec.)	(**unstable star**)
	$s_1 = s_2 < 0$		(**stable star**)
case vii	$s_1 = s_2 > 0$	(1 lin. ind. eigenvec.)	(**unstable degenerate node**)
	$s_1 = s_2 < 0$		(**stable degenerate node**)
case viii	$s_1 = s_2 = 0$	(outrageously trivial)	

Play with `equilibrium_points.m`

Cases shown in **blue** are called ***hyperbolic equilibria***.